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A Finitely Presented Solvable Group that is not Residually Finite

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The possibility that all finitely presented solvable groups be residually finite has been suggested from time to time. The purpose of this note is to show that this is not necessarily so. The basic idea in this construction is contained in [1].

Let G be the finitely presented group defined as follows

$$G=\langle a, s, t;\ s^t=s,\ a^s=a\,a^t,\ a^2=[[a, a^t], a]=[[a, a^t], s]=[[a, a^t], t]=1\rangle.$$

We claim that G is solvable (in fact center-by-metabelian) but not residually finite. To prove this let us put

$$c=[a, a^t] \quad \text{and} \quad C=\mathrm{gp}(c).$$

It is clear from the defining relations of G that C is contained in the center of G and that G/C has the presentation

$$G/C=\langle \alpha, \sigma, \tau;\ \sigma^\tau=\sigma,\ \alpha^\sigma=\alpha\,\alpha^\tau,\ \alpha^2=[\alpha, \alpha^\tau]=1\rangle$$

where here

$$\alpha=a\,C, \quad \sigma=s\,C \quad \text{and} \quad \tau=t\,C.$$

It follows, by a straightforward adaptation of the argument in [1], that G/C is metabelian. For completeness we shall sketch the proof here. Put

$$\alpha_i=\alpha^{\tau^i}\ (i=0, \pm 1, \ldots) \quad \text{and} \quad \beta_{-i}=\alpha^{\sigma^{-i}}\ (i=1, 2, \ldots).$$

It follows from the defining relations of G/C that

$$A=\mathrm{gp}(\ldots, \beta_{-3}, \beta_{-2}, \beta_{-1}; \ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots)$$

is a normal subgroup of G/C and that, modulo A, G/C is free abelian of rank two. So, in order to prove that G/C is metabelian, it suffices to prove that A is abelian. Notice that it follows from the defining relations of G/C that $\mathrm{gp}(\alpha_0, \alpha_1)$ is abelian. Suppose, inductively, that

$$\mathrm{gp}(\alpha_0, \alpha_1, \ldots, \alpha_n) \quad (n \geqq 1) \text{ is abelian}.$$

Then

$$1=[\alpha_0,\alpha_n]=[\alpha_0,\alpha_n]^{\tau}=[\alpha_0^{\tau},\alpha_n^{\tau}]=[\alpha_1,\alpha_{n+1}]\,.$$

Consequently

$$1=[\alpha_0,\alpha_n]=[\alpha_0,\alpha_n]^{\sigma}=[\alpha_0^{\sigma},\alpha_n^{\sigma}]=[\alpha_0\,\alpha_1,\alpha_n\,\alpha_{n+1}]=[\alpha_0,\alpha_{n+1}]\,.$$

Hence $\mathrm{gp}(\alpha_0,\alpha_1,\dots,\alpha_{n+1})$ is also abelian and inductively so too is $\mathrm{gp}(\alpha_0,\alpha_1,\dots,\alpha_n,\dots)$. It follows thence that $B=\mathrm{gp}(\dots,\alpha_{-1},\alpha_0,\alpha_1,\dots)$ is abelian. Now if $x,y\in\{\dots,\beta_{-2},\beta_{-1};\dots,\alpha_{-1},\alpha_0,\alpha_1,\dots\}$, we can find a positive integer m such that $x^{\sigma^m}, y^{\sigma^m}\in B$. Therefore

$$1=[x^{\sigma^m},y^{\sigma^m}]=[x^{\sigma^m},y^{\sigma^m}]^{\sigma^{-m}}=[x,y]\,.$$

So A is abelian as claimed.

The upshot of the preceding remarks is that we have proved

Lemma 1. *G is a solvable group; indeed it is center-by-metabelian.*

In order to complete the proof that the given group C has the required properties, let us put

$$a_i=a^{t^i}\ \ (i=0,\pm 1,\dots)\quad\text{and}\quad b_{-i}=a^{s^{-i}}\ \ (i=1,2,\dots).$$

Then

$$N=\mathrm{gp}(\dots,b_{-2},b_{-1};\dots,a_{-1},a_0,a_1,\dots)$$

is a nilpotent group of class at most two and $N/C\cong A$.

We shall need to prove that N is actually an infinite nilpotent group of class precisely two. To do so let $\tilde{N}$ be the group generated by

$$\dots,\tilde{b}_{-3},\tilde{b}_{-2},\tilde{b}_{-1};\dots,\tilde{a}_{-1},\tilde{a}_0,\tilde{a}_1,\dots;\tilde{c}$$

subject to the defining relations

$$\tilde{b}_i^2=\tilde{a}_k^2=\tilde{c}^2=1=[\tilde{c},\tilde{b}_i]=[\tilde{c},\tilde{a}_j],$$

and

$$[\tilde{b}_i,\tilde{b}_j]=[\tilde{b}_i,\tilde{a}_j]=[\tilde{a}_i,\tilde{a}_j]=\begin{cases}\tilde{c} & \text{if } 3\nmid j-i\\ 1 & \text{if } 3\mid j-i\end{cases}$$

where i and j range over all permissible integers. It follows readily that $\tilde{N}$ is nilpotent of class 2 with centre $\tilde{C}=\mathrm{gp}(\tilde{c})$ and that $\tilde{N}/\tilde{C}\cong A$.

Supposing $\tilde{b}_0=\tilde{a}_0$, let θ and φ be the automorphisms of $\tilde{N}$ defined as follows:

$$\tilde{c}\,\theta=\tilde{c},\tilde{b}_i\,\theta=\tilde{b}_i\tilde{b}_{i+1}\quad(i=-1,-2,\dots),\qquad \tilde{a}_j\,\theta=\tilde{a}_{j+1}\quad(j=0,\pm 1,\dots),$$

and

$$\tilde{c}\,\varphi=\tilde{c},\qquad \tilde{b}_i\,\varphi=\tilde{b}_{i+1}\quad(i=-1,-2,\dots),\qquad \tilde{a}_j\,\varphi=\tilde{a}_j\tilde{a}_{j+1}\quad(j=0,\pm 1,\dots).$$

Noting that $\theta\varphi=\varphi\theta$, we may form the split extension $\tilde{G}$ of $\tilde{N}$ by a free abelian group on $\tilde{s}$ and $\tilde{t}$ where $\tilde{s}$ induces θ and $\tilde{t}$ induces φ, via conjugation. It is not hard to see that

$$\tilde{G}=\operatorname{gp}(\tilde{s},\tilde{t},\tilde{a}),$$

where $\tilde{a}=\tilde{a}_0$. The defining relations for G are satisfied also when s,t and a are replaced, respectively, by $\tilde{s},\tilde{t}$ and $\tilde{a}$. Hence the mapping of G onto $\tilde{G}$ defined by

$$s\mapsto\tilde{s},\qquad t\mapsto\tilde{t},\qquad a\mapsto\tilde{a}$$

defines a homomorphism of G onto $\tilde{G}$. This homomorphism maps N onto $\tilde{N}$. Now $\tilde{N}$ is an infinite nilpotent group of class precisely two; it follows that N is also an infinite nilpotent group of class two.

It is easy now to deduce that G is not residually finite. In fact if K is a normal subgroup of G of finite index, then $K\cap N\neq 1$ since N is infinite. But a non-trivial normal subgroup of a nilpotent group always intersects the centre non-trivially. The centre C of N is of order 2; therefore $K\geqq C$, i.e., $c\in K$. This means every normal subgroup of G of finite index contains c. So G is not residually finite. Putting this fact together with Lemma 1 shows that G has the desired properties.

References

1. Baumslag, G.: A finitely presented metabelian group with a free abelian derived group of infinite rank. Proc. Amer. math. Soc. **35**, 61–62 (1972)

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