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A Non-free Parafree Group All of Whose Countable Subgroups are Free

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A group G is called parafree if it is residually nilpotent and if there exists a free group F and a homomorphism $\varphi: F \to G$ such that φ induces isomorphisms $\varphi_i: F/\gamma_i F \xrightarrow{\sim} G/\gamma_i G$, $i=1, 2, \ldots$ modulo the terms of the lower central series. In [B, Theorem 2.3] the first named author has given an example of a non-free parafree group which is locally free. Here we construct a non-free parafree group G with the property that all countable subgroups are free (which is *countably free*). For its construction we exploit a method due to G. Higman [H] who has given an example of a non-free group which is countably free.

We first construct an ascending system $\Phi = \{U_\mu\}$ of groups U_μ indexed by the countable ordinals μ with the following properties.

(i) U_μ is free on a countably infinite set of generators;

(ii) if $\lambda < \mu$ then U_λ is a proper subgroup of U_μ and the embedding induces an isomorphism $(U_\lambda)_{\mathrm{ab}} \cong (U_\mu)_{\mathrm{ab}}$ of the commutator factor groups;

(iii) if μ is a limit ordinal then $U_\mu = \bigcup_{\lambda<\mu} U_\lambda$;

(iv) if $\lambda < \mu$ and K is a finitely generated free factor of U_λ then K is a free factor of U_μ.

We prove the existence of such a system Φ by induction.

(a) First let $\mu = \nu + 1$. If $b_1, b_2, \ldots$ is a free generating set of U_ν, then we define $U_\mu = U_{\nu+1}$ to be free on $a_1, a_2, \ldots$ and make the identification

$$b_i = a_i [a_i, a_{i+1}] = a_i^{a_{i+1}}, \qquad i \geqq 1.$$

By definition U_μ is free and U_ν is a proper subgroup of U_μ such that $(U_\nu)_{\mathrm{ab}} \cong (U_\mu)_{\mathrm{ab}}$; hence (i) and (ii) are satisfied.

To prove (iv) let K be a finitely generated free factor of U_λ for some $\lambda < \mu = \nu + 1$. Then by induction K is a free factor of U_ν. Since K is finitely generated there exists $n \geqq 1$ such that K is contained in $B = \langle b_1, b_2, \ldots, b_n \rangle \subseteq U_\nu$, hence is a free factor of B. But for any n the set $b_1, b_2, \ldots, b_n, a_{n+1}, a_{n+2}, \ldots$ is a free generating set of U_μ. Hence K is a free factor of U_μ.

(b) Now let μ be a limit ordinal. Then by (iii) we have to set $U_\mu = \bigcup_{\lambda<\mu} U_\lambda$. It is then obvious that (ii) holds. In order to prove (i) and (iv) we need the following lemma. Let us call a subgroup K of U_μ *finitely generated basic* if it is a free factor in any finitely generated group L with $K \subseteq L \subseteq U_\mu$.

Lemma. *Any finitely generated subgroup H of U_μ is contained in a finitely generated basic subgroup K of U_μ.*

Proof. Since H is finitely generated it is contained in some U_λ, $\lambda<\mu$ and hence in a finitely generated free factor K of U_λ. Now if L is any finitely generated subgroup of U_μ with $K \subseteq L \subseteq U_\mu$ then we have $L \subseteq U_\nu$ for some $\lambda \leqq \nu < \mu$. By induction it follows from (iv) that K is a free factor of U_ν and hence of L. This proves the lemma.

We now prove (iv). Let K be a finitely generated free factor of some U_λ, $\lambda<\mu$. By induction K is a free factor in every U_ν with $\lambda \leqq \nu < \mu$, so that K is certainly a finitely generated basic subgroup of U_μ. Consider now an ascending sequence of countably many finitely generated subgroups $B_i \subseteq U_\mu$, $i=1,2,\ldots$ with $B_1=K$ and $\bigcup_{i=1}^{\infty} B_i = U_\mu$. Define $V_1=B_1$ and if V_i is constructed let V_{i+1} be a finitely generated basic subgroup containing $B_{i+1} \cup V_i$. Since V_i is free and is a free factor of V_{i+1} there exist (free) groups W_i such that

$$V_{i+1} = V_i \star W_i,$$
$$V_{i+2} = V_{i+1} \star W_{i+1} = V_i \star W_i \star W_{i+1}, \quad \text{etc.}$$

Thus $U_\mu = K \star W$ where $W = \bigstar_{i=1}^{\infty} W_i$ and K is a free factor in U_μ.

It remains to prove (i). Consider an ascending union of countably many finitely generated subgroups $B_i \subseteq U_\mu$, $i=1,2,\ldots$ with $B_1=\{e\}$ and $\bigcup_{i=1}^{\infty} B_i = U_\mu$ and define V_i and W_i as above. We obtain that $U_\mu = \bigstar_{i=1}^{\infty} W_i$; hence U_μ is free on a countable set of generators.

Theorem. *The group $G = \bigcup_{\lambda<\omega} U_\lambda$, where ω is the first uncountable ordinal, is countably free, parafree, but not free.*

In order to prove that every countable subgroup H of G is free, we note that every $h \in H$ is contained in some U_λ for some countable ordinal $\lambda = \lambda(h)$. Since the set of $\lambda(h)$ is countable there is a countable ordinal ν with $\nu > \lambda(h)$ for all $h \in G$. Thus $H \subseteq U_\nu$ and since U_ν is free, H is free.

In order to prove that G is parafree we use Corollary IV.5.5 of [St]. By definition of G and by (ii) we have that $G_{\mathrm{ab}} = \varinjlim (U_\lambda)_{\mathrm{ab}}$ is free abelian of countable rank. Also, by definition of G and (i), $H_2(G) = \varinjlim H_2(U_\lambda) = 0$. It remains to prove that G is residually nilpotent. If $w \in G$ is an element contained in every term $\gamma_k G$ of the lower central series of G, then for every $k \geqq 1$ the element w can be written as a product of k-fold commutators. There are countably many elements appearing in these descriptions of w. Consider the subgroup H of G generated by these elements. Since H is free and since $w \in \gamma_k H$ for every k, it turns out that $w = e$. Thus G is residually nilpotent and hence parafree.

In order to prove that G is not free we only have to remark that the order of G is uncountable but the order of G_{ab} is countable.

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