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A PROBLEM OF STALLINGS ON THE DIRECT SQUARE OF A FREE GROUP

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ABSTRACT. Let F be a free group of finite rank; let $\Delta(F)$ denote the diagonal subgroup of $F \times F$, and let A and B be finitely presented subgroups of $F \times F$. It is shown that $A \cap \Delta(F)$ is finitely presented, that, if neither A nor B is free, then $A \cap B$ is finitely presented, and that there are examples where both A and B are free of finite rank such that $A \cap B$ is not finitely generated.

1. Let F be a free group of finite rank. In [**5**, p. 83, Problem 6], John R. Stallings asks:

If A and B are finitely presented subgroups of the direct square $D = F \times F$ of F, is $A \cap B$ finitely generated? In particular, if $B = \Delta(F)$, the diagonal subgroup of D, is $A \cap B$ finitely generated?

2. This problem came to my attention almost three years ago. I noted then, that under the right hypotheses, the answer to Problem 6 is in the affirmative, but under different hypotheses the answer is in the negative. Problem 6 was prompted by S. M. Gersten's now well-known fixed point theorem [**2**]: *the fixed point subgroup of an automorphism of a finitely generated free group is finitely generated.* In view of the interest in this theorem and in view of the fact that no one else has ventured to answer Problem 6 in print, I have elected to do so in this short, simple note, where I shall use the notation introduced above throughout. I have taken advantage of the recent work of Goldstein and Turner to improve a little on my original observations.

3. I want to start out with two positive results. The first of these is an easy consequence of the main theorem, Theorem B of [**1**].

THEOREM 1. *If neither A nor B is free, then $A \cap B$ is finitely presented.*

PROOF. By Theorem B of [**1**] a nonfree finitely presented subgroup A of D contains a subgroup A_1 of finite index such that $A_1 = U \times V$ where U and V are subgroups of F. Similarly B contains a subgroup B_1 of finite index such that $B_1 = W \times X$ where again W and X are subgroups of F. Hence

$$A_1 \cap B_1 = (U \cap W) \times (V \cap X).$$

By a theorem of Howson [**4**], the intersection of two finitely generated subgroups of a free group is again finitely generated. So $A_1 \cap B_1$ is a direct product of two finitely generated free groups. But $A_1 \cap B_1$ is of finite index in $A \cap B$. Hence $A \cap B$ is finitely presented.

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4. On combining the ideas used in the proof of Theorem 1 with a recent theorem of Goldstein and Turner [**3**], it is straightforward to show that the more restrictive version of Problem 6 has a positive answer.

THEOREM 2. *Let A be a finitely presented subgroup of D. Then $A \cap \Delta(F)$ is finitely presented.*

PROOF. Suppose, to begin with, that A is not free. Then as in the proof of Theorem 1, there is a subgroup A_1 of finite index in A such that $A_1 = U \times V$ where U and V are (finitely generated) subgroups of F. Now

$$A_1 \cap \Delta(F) = \Delta(U) \cap \Delta(V) = \Delta(U \cap V).$$

It follows, on again invoking Howson's theorem, that $A \cap \Delta(F)$ is finitely presented.

We are left then with the case where A is free. We find it more convenient at this stage to delay the rest of the proof until we have proved Theorem 3.

5. Next we concoct some counterexamples to Problem 6. These are contained in the following simple theorem.

THEOREM 3. *Let A be a free subgroup of D of finite rank and let π_1, π_2 be the projections of A onto its first and second components respectively, and let K_2 be the kernel of π_1, K_1 the kernel of π_2. If one or other of K_1, K_2 is nontrivial, then either $A \cap (F \times 1)$ is a free group of infinite rank or else $A(\cap(1 \times F)$ is a free group of infinite rank.*

PROOF. According to Theorem B of [**1**], either $K_2 = 1$ or else $K_1 = 1$. We may assume then that $K_2 = 1$. Let $(a_1, b_1), \ldots, (a_n, b_n)$ be a finite, free set of generators of A. This set can be transformed by a finite succession of Nielsen transformations into a second set of free generators

$$(c_1, d_1), \ldots, (c_m, d_m), (c_{m+1}, 1), \ldots, (c_n, 1)$$

of A, where $m \leq n$ and $c_1, \ldots, c_m$ freely generate a free group. Since $K_2 = 1$ it follows that $c_1, \ldots, c_n$ freely generate a free group E. Moreover, by hypothesis K_1 is nontrivial. Hence $m < n$. It follows that K_1 is isomorphic to the normal closure in E of $c_{m+1}, \ldots, c_n$. But $K_1 = A \cap (F \times 1)$. This proves Theorem 3.

We are now in a position to complete the proof of Theorem 2. We adopt the notation of Theorem 3. The upshot here is that since A is free we can assume without loss of generality that A is generated by the elements $(c_1, d_1), \ldots, (c_m, d_m)$, $(c_{m+1}, 1), \ldots, (c_n, 1)$, where $m \leq n$ and $c_1, \ldots, c_n$ freely generate a free group. But then $A \cap \Delta(F)$ is isomorphic to the set of points fixed by the homomorphism of $gp(c_1, \ldots, c_n)$ into $gp(d_1, \ldots, d_m)$ defined by

$$c_1 \mapsto d_1, \ldots, c_m \mapsto d_m, c_{m+1} \mapsto 1, \ldots, c_n \mapsto 1.$$

This fixed point set is a finitely generated group by a recent result of Goldstein and Turner [**3**].

It is worth noting then that Theorem 2 provides a positive answer to the more restrictive version of Problem 6.

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