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Source: *Proceedings of the American Mathematical Society*, Vol. 108, No. 3 (Mar., 1990), pp. 633-635

Published by: American Mathematical Society

Stable URL: <http://www.jstor.org/stable/2047781>

Accessed: 13/03/2009 10:35

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A REMARK ON FINITELY PRESENTED INFINITE DIMENSIONAL ALGEBRAS

GILBERT BAUMSLAG AND PETER B. SHALEN

(Communicated by Donald S. Passman)

ABSTRACT. By estimating dimensions of representation varieties, we show that certain finitely presented algebras are infinite dimensional.

The object of this note is to use a tiny sliver of the geometry of representations of a finitely generated algebra to prove the following

Theorem. *Let A be an augmented algebra (over a field k) with augmentation ideal \mathcal{A} , given by the finite presentation*

$$A = \langle x_1, \dots, x_m; w_1^q, \dots, w_n^q \rangle \quad (w_i \in \mathcal{A}, q \geq 2).$$

If

$$n \leq (m - 1)q$$

then A is infinite dimensional.

Our theorem should be viewed in the light of the following theorem of J. Levitzki [3]: *if every element of the augmentation ideal of a finitely generated augmented algebra A is nilpotent of bounded degree, then A is finite dimensional.* It also can be viewed as a counterpart to a similar theorem about finitely presented groups that we proved a couple of years ago [1].

Before we embark on the proof of the theorem we observe first that we lose nothing if we assume that k is algebraically closed. The basic idea is to make use of the parametrisation of the set $X(A, q)$ of all the equivalence classes of semisimple representations of A in $M(q, k)$, the k -algebra of all $q \times q$ matrices over k , given by Procesi [4]. We recall the details of this parametrisation in a form suitable for the purposes we have in mind. To this end, suppose that F is the free associative k -algebra on x_1, \dots, x_m , and that $U = M(q, k)^m$. We associate to each representation ρ of F the point

$$u = (\rho(x_1), \dots, \rho(x_m)) \in U.$$

Received by the editors December 13, 1988 and, in revised form, April 17, 1989.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 16A06; Secondary 14A10, 15A24, 15A33.

The authors acknowledge support of the National Science Foundation, Grant Numbers DMS-8703251 and DMS-8701804.

Each element $w = w(x_1, \dots, x_m) \in F$ defines q polynomial functions f_w^i ($i = 0, \dots, q - 1$) on U , as follows:

$f_w^i(u)$ = the coefficient of the degree i term of the characteristic polynomial of $\rho(w)$, where ρ is the representation of F in $M(q, k)$ corresponding to u .

Then it turns out that the k -subalgebra B of F generated by these polynomial functions f_w^i , where w ranges over F and $i = 0, \dots, q - 1$, is a finitely generated subalgebra of the k -algebra P of all polynomial functions on U . Notice that P is the k -algebra of polynomials in mq^2 variables. Now let $X(F, q)$ be the affine algebraic set corresponding to this algebra B and let p be the canonical map from U into $X(F, q)$. Then Procesi [4] proves that the following hold:

- (1) p is onto $X(F, q)$;
- (2) if S is the subset of U consisting of semi-simple representations of F , then p maps S onto $X(F, q)$;
- (3) if $\rho \in S$ is irreducible, then $p^{-1}(p(\rho))$ is the set of all representations of F in $M(m, q)$ equivalent to ρ .

It follows, in particular, that if ρ is an irreducible representation of F , then $p^{-1}(\rho)$ is of dimension $q^2 - 1$. Now $X(F, 2)$ is an affine variety, i.e. it is irreducible. So it follows (see e.g. Humphreys [2, page 30]) that

$$\begin{aligned} \dim(X(F, q)) &\geq \dim(R(F, q)) - \dim(p^{-1}(\rho)) \\ &= mq^2 - (q^2 - 1) = (m - 1)q^2 + 1. \end{aligned}$$

Now if M is a matrix of degree q over k , then $M^q = 0$ if and only if its characteristic polynomial is t^q . This means that the coefficients of all of the powers of t except for t^q in the characteristic polynomial of M are zero. We need to apply this remark to the defining relations of A . Observe then that $\rho(w_j^q) = 0$ for every representation ρ of A in $M(q, k)$ if and only if the functions $f_{w_j}^i = 0$, for $i = 0, \dots, q - 1$. The existence of at least one such representation is guaranteed by the hypothesis since A is an augmented algebra. Consider then these functions $f_{w_j}^i = g_j^i$ ($i = 0, \dots, q - 1, j = 1, \dots, n$). Every such function g_j^i lies in the algebra B . Consequently they can be viewed as polynomial functions on $X(F, q)$ with values in k . Let $h_j^i = p \circ f_j^i$. Consider

$$V = \bigcap_{j,i} (h_j^i)^{-1}(0).$$

This is therefore an affine algebraic set, since the h_j^i are polynomial functions. And

$$p(V) = \bigcap_{j,i} (g_j^i)^{-1}(0)$$

is therefore also an affine algebraic set in $X(F, d)$ whose dimension is then bounded as follows:

$$\dim p(V) \geq (m-1)q^2 + 1 - nq.$$

Now every point in $p(V)$ corresponds to an equivalence class of semi-simple representations of F such that for each representation $\rho \in V$ the characteristic polynomial of every $\rho(w_j)$ is t^q . So by the remark above, ρ factors through A , yielding a semi-simple representation of A . This means that $p(V)$ parametrises a set of inequivalent semi-simple representations of A . Since a finite dimensional algebra has only finitely many inequivalent representations overall, if $\dim p(V) > 0$ then A is infinite dimensional. This is precisely what the inequality given in the theorem ensures.

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