A REMARK ON GROUPS WITH TRIVIAL MULTIPLICATOR.

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1. Let $H_n(G)$ denote the nth homology group of G (with trivial integral coefficients). The (Schur) multiplicator of G is then, by definition, $H_2(G)$.

The object of this note is to construct infinitely many 4-generator groups which are not finitely related but have trivial multiplicator. These examples should be compared with the first ones given in [1] and [2].

The basic idea of the construction is the use of a Mayer-Vietoris sequence, in a manner which is very similar to that adopted by Stallings in [3], but with a different end in view. Thus our procedure here is in direct contrast to that followed in [1] and [2].

2. If G is a group we denote the commutator subgroup of G by G', the least normal subgroup of G containing the element $g(\in G)$ by $gp_G(g)$, the conjugate $h^{-1}gh$ of g by h by g^h, and the commutator $g^{-1}h^{-1}gh$ by $[g,h]$.

Let $G=\{A*B;\ C\}$ be the generalized free product of its subgroups A and B amalgamating C. Then there is an exact sequence (called the Mayer-Vietoris sequence for the generalized free product G) of the form

$$\begin{aligned}\cdots \to H_n(A)\oplus H_n(B)\to H_n(G)\to H_{n-1}(C)\to\cdots\\ \to H_2(A)\oplus H_2(B)\to H_2(G)\to H_1(C)\\ \to H_1(A)+H_1(B)\to H_1(G)\end{aligned}$$

(see [3] and the references cited there).

3. The idea involved in the construction is as follows. Let A and B be groups with trivial multiplicator and let C be a common subgroup of both A and B with $H_1(C)=0$. Notice that $H_1(C)=C/C'$; so the condition on C is that it be perfect (i.e., coincide with its derived group). Consider the following portion of the Mayer-Vietoris sequence for $G=\{A*B;C\}$:

$$H_2(A)\oplus H_2(B)\to H_2(G)\to H_1(C).$$

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Now, by assumption, $H_2(A) = H_2(B) = H_1(C) = 0$. Therefore $H_2(G) = 0$. If both A and B are finitely generated but C is not, then G is a finitely generated but not finitely related group in view of [4]; thus G is a finitely generated group with trivial multiplicator which is not finitely related.

4. In order to explicitly construct such groups G, let A_n be the 1-relator group defined as follows:

$$A_n = \langle a, b; a^n = [a^n, a^{nb}] \rangle.$$

It is easy to see, using the basic breakdown of 1-relator groups as detailed, for example, in Magnus, Karrass, and Solitar [5], that

$$C_n = gp_{A_n}(a^n)$$

is not finitely generated and that C_n is perfect. By a theorem of Lyndon [6], $H_2(A_n) = 0$ (because $a^{-n}[a^n, a^{nb}]$ is not in the derived group of the free group on a and b). Now let $G_n = \{A_n * A_n; C_n\}$. Then it follows from the remarks in Section 3 that G_n is a finitely generated, but not finitely presented, group with trivial multiplicator. Notice that the groups G_n can all be generated by four elements and that $G_n / G'_n \cong \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}_n \times \mathbf{Z}_n$ where $\mathbf{Z}_n$ denotes a cyclic group of order n and $\mathbf{Z}$ an infinite cyclic group. Hence $G_n \cong G_m$ only if $n = m$. It follows that we have constructed infinitely many 4-generator, infinitely related groups with trivial multiplicator.

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