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A Remark on the Multiplicative Group of the Division Ring of a Poly-Infinite-Cyclic Group*

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For my friend and colleague Wilhelm Magnus on his 65th birthday†

§1. L. Hua [1] has proved that the multiplicative group of a division ring is solvable only if it is commutative. In this regard Nathan Jacobson remarks in his book ([2], p. 191) that very little is known about the multiplicative group of a division ring (see also W. R. Scott [3]). The objective of this note is to add to this knowledge.

In order to explain how, we recall that a group G is termed poly-infinite-cyclic if there exists a finite series

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_k = G \tag{1}$$

of subgroups of G, each normal in the succeeding, with every factor group G_{i+1}/G_i, $i = 0, 1, \cdots, k-1$, infinite cyclic. The integral group ring $\mathbb{Z}G$ of such a poly-infinite-cyclic group G is a left (and right) Ore domain (O. Ore [4] and see also P. M. Cohn [5], p. 23) and hence can be embedded in a unique division ring, D, which we shall refer to as the division ring of fractions of G (see again [4] or [5], p. 23).

The purpose of this note is to prove that the multiplicative group D^* of D inherits some of the solvability of the group G by virtue of the following

THEOREM. *Let G be poly-infinite-cyclic and let D be its division ring of fractions. Then D^* is hypoabelian, i.e., its (possibly transfinite) derived series terminates in the identity.*

The proof of this theorem is essentially a combination of an idea of Ian Hughes [6] with one of W. Magnus [7].

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§2. Suppose that C is a division ring and that τ is an automorphism of C. Let H be the so-called Hilbert division ring of Laurent series in an indeterminate t over C, where t induces the given automorphism τ in C by conjugation (cf. [2], p. 187). Thus, by definition,

$$H = \left\{ \sum_{i \geqq l} c_i t^i \,\middle|\, c_i \in C, \, l \text{ any integer} \right\},$$

with the obvious definitions of equality, addition and multiplication. We emphasize that each element of H involves only finitely many negative powers of t and that if $c \in C$, then

$$tc = c^{\tau} t .$$

The following lemma is the key to the proof of our theorem.

Lemma. *If C^* is hypoabelian, then so too is H^*.*

Proof: Put

$$M = \left\{ \sum_{i=0}^{\infty} c_i t^i \,\middle|\, c_0 = 1 \right\}, \qquad N = gp(C^*, t) .$$

Then it is easy to see that

$$M \trianglelefteq H^*, \qquad M \cap N = 1 \qquad \text{and} \qquad H^* = NM . \tag{2}$$

Now it follows from (2) that

$$H^*/M \cong N . \tag{3}$$

Since an extension of one hypoabelian group by another is again hypoabelian, it suffices by (3) to prove that both M and N are hypoabelian.

To do so first observe that $C^* \trianglelefteq N$ and that N/C^* is infinite cyclic (on tC^*). Hence N is hypoabelian.

Finally we need only point out that, if

$$x = \sum_{i=0}^{\infty} c_i t^i \in M^{(i)},$$

the i-th term of the derived series of M, then it follows by a direct computation (cf. W. Magnus [7]) that

$$c_0 = 1 \qquad \text{and} \qquad c_1 = c_2 = \cdots = c_i = 0 .$$

Therefore,

$$\bigcap_{i=1}^{\infty} M^{(i)} = 1$$

and M is also hypoabelian as required. This completes the proof of the lemma.

§3. We are now in a position to prove the theorem. Thus let G be poly-infinite-cyclic and let D be its division ring of fractions. The proof is by induction on the length k of a poly-infinite-cyclic series (1) for G.

If $k = 1$, G is cyclic and so its division ring of fractions is commutative; hence in this case D^* is obviously hypoabelian.

Thus we may suppose $k > 1$. Put $F = G_{k-1}$ and let t be chosen so that

$$G = gp(F, t) .$$

Notice that F has a series of the type (1) of length $k - 1$. So if C is its division ring of fractions, C^* is hypoabelian by the induction hypothesis.

Now $F \trianglelefteq G$. Therefore t induces an automorphism of $\mathbb{Z}F$ and thus, by the uniqueness of C, it also induces an automorphism, say τ, of C. Hence we can form H, the Hilbert division ring of Laurent series in t over C (with t inducing τ in C). Since $\mathbb{Z}G$ is readily seen to be a subring of H (cf. Ian Hughes [6]), it follows again from the uniqueness of the division ring of fractions of G that D is itself (isomorphic to) a subdivision ring of H. But H^* is hypoabelian by the lemma; hence D^* is too.

It is perhaps worth noting that it follows from the proof of the theorem that if G is poly-infinite-cyclic of length k, then the derived series of D^* terminates in the identity in ωk steps.

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