

Direct Decompositions of Finitely Generated Torsion-Free Nilpotent Groups

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§ 1. Introduction

A subgroup H of a group G is termed isolated in G if for every positive integer n , $g \in H$ whenever $g^n \in H$ ($g \in G$). It is easy to show that if G is a finitely generated torsion-free nilpotent group, then G satisfies the maximum condition for isolated subgroups, and can therefore be expressed as a direct product of a finite number of directly indecomposable groups. The classical Krull-Remak-Schmidt theorem asserts that under certain conditions two decompositions of a group into a direct product of directly indecomposable groups are isomorphic in the sense that the factors in the two decompositions can be matched up so that corresponding factors are isomorphic (cf. e.g., Kurosh [4], vol. 2, p. 81).

The main purpose of this note is to show that the conclusion of the Krull-Remak-Schmidt theorem is not valid for finitely generated torsion-free nilpotent groups. In fact we shall prove

Theorem 1. *Let m and n be given integers with $m > 1$, $n > 1$. Then there exists a finitely generated torsion-free nilpotent group which can be expressed as a direct product of m directly indecomposable groups and also as a direct product of n directly indecomposable groups in such a way that no factor in the first decomposition is isomorphic to a factor in the second.*

Theorem 1 raises the question as to whether there are any invariants involving direct products of finitely generated torsion-free nilpotent groups. In fact, in a certain sense, there are. In order to explain, let us denote by $r(G)$ the rational hull (or Mal'cev completion) of G , i.e., the unique (up to isomorphism) minimal torsion-free nilpotent group containing G in which extraction of n -th roots is possible for every non-zero integer n (Mal'cev [5]). We term G *rationally indecomposable* if $r(G)$ is directly indecomposable. Then it is easy to prove

Theorem 2. *Let G be a finitely generated torsion-free nilpotent group. Then the number of factors in any direct decomposition of G into rationally indecomposable groups is an invariant of G .*

We have been unable to determine whether an infinite cyclic group can be cancelled in two direct decompositions of a finitely generated torsion-free nilpotent group (it cannot always be cancelled if there is torsion – see Baumslag [2]). The best that we can do is to prove the (again almost obvious)

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Theorem 3. *Suppose that A and B are rationally indecomposable finitely generated torsion-free nilpotent groups and that C is infinite cyclic. If*

$$A \times C \cong B \times C$$

then $A \cong B$.

The organization of the rest of this paper is as follows. In § 2 we introduce some of the ideas and techniques to be used later on and, additionally, we describe some related results. We prove Theorem 2 in § 3, Theorem 3 in § 4 and, finally, Theorem 1 in § 5.

§ 2. Preliminaries

We term a group G a \mathcal{D} -group or a *rational group* if extraction of n -th roots is uniquely possible in G for every positive integer n . A subgroup H of a \mathcal{D} -group G is called a \mathcal{D} -subgroup or a *rational subgroup* if H is itself a \mathcal{D} -group. If G is a rational nilpotent group then G/G' is a rational abelian group, where G' denotes the derived group of G (see Baumslag [1]). Thus G/G' may be viewed as a rational vector space. If this vector space is finite dimensional, then we say that the rational nilpotent group G is of *finite rank*. It is not hard to prove that if G is a rational nilpotent group of finite rank then $\gamma_n G/\gamma_{n+1} G$ is a rational abelian group and, viewed as a rational vector space, is finite dimensional for every n . (This explains the terminology “finite rank”.) It follows that a rational nilpotent group of finite rank satisfies the maximal and minimal conditions for rational subgroups. Hence it is easy to prove (cf. e.g., Jacobson [3], p. 12) the following

Theorem A. *If G is a rational nilpotent group of finite rank, then any two decompositions of G into a direct product of directly indecomposable groups are isomorphic.*

By making use of Theorem A it is not hard to prove the following two theorems.

Theorem B. *Let H be a finitely generated torsion-free nilpotent group. Let $r_p(H)$ denote the subgroup of $r(H)$ generated by H and its q^m -th roots ($m=1, 2, \dots$), with q ranging over all primes different from the prime p . Then H is rationally indecomposable if and only if $r_p(H)$ is directly indecomposable for every p .*

Theorem C. *Let H be a finitely generated torsion-free nilpotent group with a normal endomorphism η such that*

$$1 < \text{kernel of } \eta^n < H \quad \text{for } n=1, 2, \dots$$

Then there is a subgroup K of finite index in H which is decomposable into a non-trivial direct product.

We shall have no need of Theorems B and C here and so we omit the proofs.

It is worth noting that Theorem C is an analogue of Fitting’s Lemma (cf. e.g., Jacobson [3], p. 9). The hypothesis of Theorem C is not strong enough to be able to draw the conclusion that H is itself a non-trivial direct product. Indeed, let H

be the following finitely presented group:

$$H = \langle a, a_1, a_2, a_3, b, c; a_1^a = a_1 a_2, a_2^a = a_2 a_3^2, a_3^a = a_3, [a_i, a_j] = [a_i, b] \\ = [c, a_i] = 1 (i, j = 1, 2, 3), c^2 = a_2 b, c^a = c a_3 \rangle$$

(here $x^y = y^{-1} x y$, $[x, y] = x^{-1} y^{-1} x y$). It turns out that H is a finitely generated torsion-free nilpotent group of class three which is directly indecomposable. However the normal endomorphism η of H defined by

$$\eta: a \mapsto a, a_i \mapsto a_i^2 (i = 1, 2, 3), b \mapsto 1, c \mapsto a_2$$

satisfies the hypothesis of Theorem C. Again we omit the proofs of these remarks.

§ 3. The Proof of Theorem 2

Suppose H is a finitely generated torsion-free nilpotent group and that H is decomposed in two ways as a direct product of rationally indecomposable groups:

$$H_1 \times \cdots \times H_m = H = K_1 \times \cdots \times K_n. \quad (1)$$

It follows from (1) that

$$r(H_1) \times \cdots \times r(H_m) \cong r(H) \cong r(K_1) \times \cdots \times r(K_n). \quad (2)$$

Now $r(H)$ is a rational nilpotent group of finite rank. Moreover, by hypothesis the factors involved in (2) are indecomposable rational nilpotent groups. Hence Theorem A applies and so $m = n$, as required.

§ 4. The Proof of Theorem 3

We shall need the following

Lemma 1. *Let H be a finitely generated torsion-free nilpotent group. If H is rationally indecomposable then $Z/H' \cap Z$ is finite, where here Z is the center of H .*

Proof. Suppose that $Z/H' \cap Z$ is infinite. Then there exists an element $a \in Z$ of infinite order modulo H' . Since $r(H)' = r(H')$ and $r(H')$ consists of those elements of H with a non-trivial power in H' (see e.g., Kurosh [4], vol. 2, p. 249), $a \notin r(H)'$. Let $A = r(\text{gp}(a))$; then $A \cap r(H)' = 1$. Since $r(H)/r(H)'$ may be viewed as a rational vector space we can choose elements $x_1, \dots, x_q \in r(H)$ such that x_1, \dots, x_q, a constitute a basis for $r(H)$ modulo $r(H)'$. Let $X = r(\text{gp}(x_1, \dots, x_q))$. Then, modulo $r(H)'$, $X \cap A \cong 1$, i.e., $X \cap A \leq r(H)'$. Thus

$$X \cap A \leq r(H)' \cap A = 1.$$

Observe also that $r(\text{gp}(X, A)) = r(H)$ since, modulo $r(H)'$, $X \cup A$ spans $r(H)$ (Steinberg [6]). Now A is central in $r(H)$ (see Kurosh [4], vol. 2, p. 244); so it follows that $\text{gp}(X, A) = X \times A$ and hence

$$r(H) = r(\text{gp}(X, A)) = r(X \times A) = r(X) \times r(A) = X \times A.$$

Therefore $r(H)$ is rationally decomposable; this contradicts the hypothesis and so completes the proof of Lemma 1.

It is easy now to deduce Theorem 3. Thus suppose that G is a finitely generated torsion-free nilpotent group such that

$$A \times C = G = B \times D \quad (3)$$

where $C = \text{gp}(x)$ and $D = \text{gp}(d)$ are infinite cyclic and A is rationally indecomposable. Let U be the center of A and let V be the center of B . By Lemma 1 $U/U \cap A'$ is finite. So, in view of (3), $V/V \cap B'$ is also finite. Now observe that, again making use of (3),

$$c = bd^m \quad (b \in B). \quad (4)$$

Since c is central in G , so is b and hence $b \in V$. Therefore $b^l \in B'$ for some $l \geq 1$. Putting this together with (4) yields

$$c^l \equiv d^{lm}(G').$$

It is however clear from (3) that c^l is an lm -th power modulo G' only if $lm = \pm l$, i.e., if $m = \pm 1$. Thus $c = bd^{\pm 1}$ (by (4)). This implies that $\text{gp}(c) = \text{gp}(bd^{\pm 1})$ meets B trivially. So $G = \text{gp}(c, B) = C \times B$. On factoring out C we find

$$A \cong (A \times C)/C = G/C = (C \times B)/C \cong B$$

as required.

In view of Theorem 3 it is perhaps worth noting the following criterion for a finitely generated torsion-free nilpotent group to be rationally indecomposable.

Lemma 2. *A torsion-free nilpotent group G with a cyclic center is rationally indecomposable.*

Proof. If Z is the center of G , then the center of $r(G)$ is $r(Z)$, a multiplicative copy of the rationals (Kurosh [4], vol. 2, p. 249). So $r(G)$ is directly indecomposable since a non-trivial nilpotent group has a non-trivial center.

§ 5. The Proof of Theorem 1

The proof of Theorem 1 centers around a family of torsion-free nilpotent groups which depend on two parameters, a positive integer n and a prime p . For clarity these groups will be built in three stages as detailed below.

To begin with let A be a free abelian group of rank $3n$ on $a_1, b_1, c_1, \dots, a_n, b_n, c_n$. The map

$$a_i \mapsto a_i b_i, \quad b_i \mapsto b_i c_i, \quad c_i \mapsto c_i \quad (i = 1, \dots, n)$$

defines an automorphism τ of A of infinite order. Let $T = \langle t \rangle$ be the infinite cyclic group on t and let B be the split extension of A by T with t inducing the automorphism τ on A :

$$B = A] T. \quad (5)$$

It is easy to see that A is a torsion-free nilpotent group of class 3.

Next we form the direct product D of B and a free abelian group F of rank n on f_1, f_2, \dots, f_n :

$$D = B \times F. \quad (6)$$

Of course D is also a torsion-free nilpotent group of class 3.

The desired group $G = G(n, p)$ is the subgroup of the rational hull $r(D)$ of D which is defined as follows:

$$G = \text{gp}(D, s_1, s_2, \dots, s_n) \quad (7)$$

where s_i is the (unique) p -th root of $b_i f_i$:

$$s_i^p = b_i f_i \quad (i = 1, \dots, n).$$

Notice that G is also a torsion-free nilpotent group of class 3.

The proof of Theorem 1 involves the demonstration that these groups $G(n, p)$ have certain properties. The first of these is given in

Lemma 3. $G = G(n, p)$ is directly indecomposable.

Proof. Suppose G is directly decomposable, say

$$G = X \times Y.$$

Our first objective is to prove that either X or Y is abelian. Indeed, let us suppose that neither X nor Y is abelian. Then going over to the rational hull $r(G)$ of G we find

$$r(G) = r(X) \times r(Y) \quad (8)$$

is the direct product of two non-abelian groups $r(X)$ and $r(Y)$. On the other hand, it follows from (6) that

$$r(G) = r(B) \times r(F). \quad (9)$$

Putting (8) and (9) together with Theorem A it follows that $r(B)$ is a direct product of two non-abelian groups, say

$$r(B) = U \times V. \quad (10)$$

Now observe that $r(B)$ has an abelian normal subgroup $r(A)$ such that $r(B)/r(A)$ is isomorphic to a (multiplicative) copy of the additive group Q of rational numbers (see (5)). It is easy to see that

$$r(A) = \text{cr}(r(B)'; r(B)),$$

where if X is a subgroup of the group Y we denote the centraliser of X in Y by $\text{cr}(X; Y)$. Now

$$\text{cr}(r(B)'; r(B)) = \text{cr}(U'; U) \times \text{cr}(V'; V).$$

But as we observed above $r(B)/r(A) \cong Q$. Hence either

$$\text{cr}(U'; U) = U \quad \text{or} \quad \text{cr}(V'; V) = V.$$

Suppose that the first possibility holds. Then

$$U \cong r(A)$$

is abelian, a contradiction. Similarly the other case can also be disposed of.

So if G is directly decomposable it can be written in the form

$$G = X \times Y \tag{11}$$

with Y infinite cyclic:

$$Y = \langle y \rangle. \tag{12}$$

Of course, y is central in G . Now if we put $e_i = [t, s_i^{-1}]$ then it is not hard to check that $e_i^p = c_i$ and that the center Z of G is generated by $e_1, \dots, e_n, f_1, \dots, f_n$:

$$Z = \text{gp}(e_1, \dots, e_n, f_1, \dots, f_n). \tag{13}$$

So

$$y = e_1^{l_1} \dots e_n^{l_n} f_1^{m_1} \dots f_n^{m_n}. \tag{14}$$

But now, working modulo G' , we see from (14) that

$$y \equiv f_1^{m_1} \dots f_n^{m_n} \equiv (s_1^{m_1} \dots s_n^{m_n})^p$$

since $f_i \equiv s_i^p$ modulo G' . In view of (11), however, this is impossible since, modulo G' , y generates an infinite cyclic direct factor of G .

This completes the proof of Lemma 3.

Now let us denote the infinite cyclic group by C and let C^l denote the direct product of l copies of C . The next step in the proof of Theorem 1 is

Lemma 4. *Let $n \geq 2$ be preassigned. Then, if p and q are distinct primes,*

$$G(n-1, p) \times G(n-1, q) \cong C^{n-1} \times S$$

where S is directly indecomposable and is not isomorphic to either $G(n-1, p)$ or $G(n-1, q)$.

Proof. We use Latin letters for the generators of $G(n-1, p)$ as we did initially and Greek letters for those of $G(n-1, q)$. Thus

$$G(n-1, q) = \text{gp}(\alpha_1, \beta_1, \gamma_1, \dots, \alpha_{n-1}, \beta_{n-1}, \gamma_{n-1}, \varphi_1, \dots, \varphi_{n-1}, \sigma_1, \dots, \sigma_{n-1})$$

with the given generators subject to the obvious relations.

Now let us put

$$T = \text{gp}(f_1 \varphi_1^{-1}, \dots, f_{n-1} \varphi_{n-1}^{-1}).$$

Then T is free abelian of rank $n-1$; moreover T is central in

$$D = G(n-1, p) \times G(n-1, q).$$

Indeed we claim T is a direct factor of D . To see how this comes about, we proceed as follows. First, noting that p and q are distinct primes, we can choose integers l

and m such that

$$lp + mq = 1.$$

Next we put

$$S = \text{gp}(t, a_1, \dots, a_{n-1}, \tau, \alpha_1, \dots, \alpha_{n-1}, \sigma_1^l s_1^m, \dots, \sigma_{n-1}^l s_{n-1}^m).$$

Notice that

$$D = \text{gp}(S, T).$$

To prove this it suffices to show that D is generated by $S \cup T$ modulo D' . Now, modulo D' ,

$$s_i^p \equiv f_i \equiv \varphi_i \equiv \sigma_i^q \quad (i = 1, \dots, n-1).$$

Hence, still working modulo D' ,

$$s_i \equiv s_i^{lp+mq} \equiv (s_i^p)^l (s_i^q)^m \equiv (\sigma_i^q)^l (s_i^q)^m \equiv (\sigma_i^l s_i^m)^q$$

and similarly

$$\sigma_i \equiv (\sigma_i^l s_i^m)^p \quad (i = 1, 2, \dots, n-1).$$

It follows that $s_i, \sigma_i \in (\text{gp}(S, T))D'$ and hence that $D = \text{gp}(S, T)$, as required, since a set which generates a nilpotent group modulo its derived group actually generates that group.

Finally, we note that $T \cap D' = 1$ and that, modulo D' , D is free abelian with basis

$$t, a_1, \dots, a_{n-1}, \tau, \alpha_1, \dots, \alpha_{n-1}, \sigma_1^l s_1^m, \dots, \sigma_{n-1}^l s_{n-1}^m, f_1 \varphi_1^{-1}, \dots, f_{n-1} \varphi_{n-1}^{-1}.$$

It follows that

$$D = S \times T$$

as required.

Next we check that S is directly indecomposable.

Notice at the outset that S is a central product of $G(n-1, p)$ and $G(n-1, q)$:

$$S = \{G(n-1, p) \times G(n-1, q); f_1 = \varphi_1, \dots, f_{n-1} = \varphi_{n-1}\}.$$

We check first that S does not have a non-trivial cyclic direct factor $\langle u \rangle$, say. Indeed if $\langle u \rangle$ is such a direct factor, then u is certainly central in S . Now it is not hard to see that if Z_1 is the center of $G(n-1, p)$ and Z_2 is the center of $G(n-1, q)$, then the center Z of S is given by

$$Z = \text{gp}(Z_1, Z_2) = \text{gp}(e_1, \dots, e_{n-1}, \varepsilon_1, \dots, \varepsilon_{n-1}, f_1, \dots, f_{n-1}),$$

where $e_i = [t, s_i^{-1}]$ and $\varepsilon_i = [\tau, \sigma_i^{-1}]$. Hence

$$u \equiv e_1^{k_1} \dots e_{n-1}^{k_{n-1}} \varepsilon_1^{l_1} \dots \varepsilon_{n-1}^{l_{n-1}} f_1^{m_1} \dots f_{n-1}^{m_{n-1}}.$$

Now, modulo S' , $s_i^p = f_i$. Hence, again working modulo S' ,

$$u \equiv f_1^{m_1} \dots f_{n-1}^{m_{n-1}} \equiv (s_1^{m_1} \dots s_{n-1}^{m_{n-1}})^p.$$

But $p > 1$; so $f_1^{m_1} \dots f_{n-1}^{m_{n-1}} = 1$ which means $u \in S'$. Consequently $u = 1$, a contradiction.

Thus if S is directly decomposable, $S = U \times V$ where neither U nor V is abelian. Put

$$0 = \text{cr}(S'; S).$$

It is easy to check that 0 is abelian and that S/C is free abelian on $t0$ and $\tau 0$. Consequently it follows that

$$U \ni u = t^i \tau^j c, \quad V \ni v = t^k \tau^l d \quad (c, d \in 0),$$

where $il - jk = \pm 1$. Now if $ij \neq 0$ then

$$\text{cr}(t^i \tau^j c; S) = \text{gp}(t, \tau, Z);$$

notice that this subgroup is abelian. Since $u = t^i \tau^j c \in U$, and $V \leq \text{cr}(u; S)$, $ij = 0$ is the only possibility because V is non-abelian. Similarly, since $v = t^k \tau^l d \in V$ and U is non-abelian, $kl = 0$. It follows readily that we may assume, without loss of generality, that

$$U \ni u = tc, \quad V \ni v = \tau d \quad (c, d \in 0).$$

Now $U \trianglelefteq S$; so

$$[u, a_i^{-1}] = b_i \in U \quad (i = 1, \dots, n-1).$$

Hence

$$[u, b_i^{-1}] = c_i \in U \quad (i = 1, \dots, n-1).$$

Similarly

$$\beta_i, \gamma_i \in V \quad (i = 1, \dots, n-1).$$

Next observe that

$$s_i = u_i v_i \quad (u_i \in U, v_i \in V);$$

since $s_i \in 0 = \text{cr}(S'; S)$, $u_i \in 0$, $v_i \in 0$. Moreover $[u_i, \tau d] = 1 = [v_i, tc]$. It follows that

$$u_i = s_i y_i \quad \text{where } y_i \in F = \text{gp}(f_1, \dots, f_{n-1}).$$

Consequently

$$(s_i y_i)^p = s_i^p y_i^p = b_i f_i y_i^p \in U.$$

This means that $f_i y_i^p \in U$ ($i = 1, \dots, n-1$). Now the elements $f_1 y_1^p, \dots, f_{n-1} y_{n-1}^p$ generate F modulo F^p and therefore $\text{gp}(f_1 y_1^p, \dots, f_{n-1} y_{n-1}^p)$ is of finite index in F . Thus it follows that

$$U \cap F \quad \text{is of finite index in } F.$$

An analogous argument applied to the elements σ_i yields

$$V \cap F \quad \text{is of finite index in } F.$$

Since F is infinite this implies $U \cap V \neq 1$, a contradiction.

Finally, in order to complete the proof of Lemma 4 we need to check that neither $G(n-1, p)$ nor $G(n-1, q)$ is isomorphic to S . This however follows immediately on noting that the rank of S is greater than the rank of $G(n-1, p)$ as well as the rank of $G(n-1, q)$.

An argument analogous to that used in the proof of Lemma 4, which we omit here, yields

Lemma 5. *Let $n \geq 3$ and let p, q and r be distinct primes. Then*

$$G(n-1, p) \times G(n-2, q) \times G(1, r) \cong C^{n-1} \times S$$

where S is directly indecomposable and is not isomorphic to any of the factors $G(n-1, p)$, $G(n-2, q)$ or $G(1, r)$.

We are now in a position to prove Theorem 1. We may assume without loss of generality, that $n \geq m$.

First we consider the case when m is even, say $m = 2l$. Pick two distinct primes p and q . Notice that if $m = 2$, Lemma 4 provides two direct decompositions of the required kind. Thus we may assume $m > 2$. Then

$$\begin{aligned} & (G(1, p) \times G(1, q))^{l-1} \times G(n+1-m, p) \times G(n+1-m, q) \\ & \cong (C \times S_1)^{l-1} \times C^{n+1-m} \times S_2 \end{aligned} \quad (15)$$

where S_1 and S_2 are the obvious central products that arise on appealing to Lemma 4. Since the right-hand side of (15) involves $2(l-1) + n + 1 - m + 1 = n$ factors we have produced a direct decomposition of the required kind.

Finally suppose $m > 1$ is odd. Then, by Lemma 5, it suffices to consider the case $m > 3$. So

$$m = 3 + m_1, \quad m_1 \text{ even.}$$

Now if p, q and r are distinct primes and if $s > 2$ is any integer, then by Lemma 5

$$G(s-1, p) \times G(s-2, q) \times G(1, r) \cong C^{s-1} \times S_1.$$

So if we take $s = n - m_1$ then it follows, on putting $m_1 = 2l_1$, that

$$\begin{aligned} & G(s-1, p) \times G(s-2, q) \times G(1, r) \times (G(1, p) \times G(1, q))^{l_1} \\ & \cong C^{n-m_1-1} \times S_1 \times (C \times S_2)^{l_1}. \end{aligned} \quad (16)$$

The right-hand-side of (16) comprises $n - m_1 - 1 + 1 + 2l_1 = n$ factors as required. This completes the proof of Theorem 1.

Bibliography

1. Baumslag, G.: Some aspects of groups with unique roots, Acta math. **104**, 217-303 (1960)
2. Baumslag, G.: Some metacyclic groups with the same finite images, Compositio Math. **29**, 249-252 (1974)

3. Jacobson, N.: The theory of rings, Mathematical Surveys II. New York: American Mathematical Society 1943
4. Kurosh, A.G.: The theory of groups, vols. 1 and 2, Translated and edited by K.A. Hirsch, New York: Chelsea 1955
5. Mal'cev, A.I.: On a class of homogeneous spaces, *Izvestija Akad. Nauk SSSR Ser. mat.* **13**, 9-32 (1949)
6. Steinberg, A.: On free nilpotent quotient groups, *Math. Z.* **85**, 185-196 (1964)

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Note Added in Print: A.L. Smelkin has informed me that one of his students has proved that the Krull-Remak-Schmidt theorem holds for finitely generated torsion-free nilpotent groups of class two. It should be noted that the examples constructed in this paper all are nilpotent of class three.