

Finitely generated cyclic extensions of free groups are residually finite

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We establish the result that a finitely generated cyclic extension of a free group is residually finite. This is done, in part, by making use of the fact that a finitely generated module over a principal ideal domain is a direct sum of cyclic modules.

1. Introduction

The purpose of this note, as the title suggests, is to prove the following

THEOREM. *A finitely generated cyclic extension of a free group is residually finite.*

There are a host of finitely generated groups with a single defining relation to which this theorem applies. These groups include the fundamental groups of two-dimensional surfaces (both orientable and non-orientable) as well as the groups

$$\text{gp}(a, b, c : c^n = [a, b]) \quad (n = 1, 2, \dots).$$

The residual finiteness of these groups is well-known (see for example, [1], [2], [4] and [5]). However the proof of the theorem provides essentially new information, even about these groups. An explicit example of a one-relator group which was not known to be residually finite until

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now is the group

$$G = \text{gp}\left\{a, b, c : [a^2, b][c, a^{-1}ba]\right\}.$$

It is not difficult to show that a finitely generated group G which is an infinite cyclic extension of a free group is not always a one-relator group. So our theorem has meaning apart from groups with a single defining relation, even in this case.

2. Some useful lemmas

The proof of the theorem depends in large measure on the following lemmas.

LEMMA 1. *A finitely generated module over a principal ideal domain R is a direct sum of a finite number of cyclic modules.*

Lemma 1 is a celebrated classical theorem (see for example [8], p. 86 for a proof). We shall apply it in the case where R is the group algebra of an infinite cyclic group over a field of p elements (p a prime).

The second result we shall need (and which can be proved directly without too much difficulty (cf. for example [3])) is

LEMMA 2. *Let \underline{V} be a nilpotent variety of prime exponent. Furthermore let F be a free group in \underline{V} . Then any set of elements of F which are independent modulo the derived group F' of F freely generate a free group in \underline{V} .*

Finally we shall make use of

LEMMA 3. *Let F be a free group and let $f \in F$ ($f \neq 1$). Then there exists a nilpotent variety \underline{V} of prime exponent such that $f \notin V(F)$ (where, as usual, $V(F)$ is the unique minimal normal subgroup of F satisfying $F/V(F) \in \underline{V}$).*

The proof of Lemma 3 is a consequence of two theorems, one due to Magnus [9], the other due to Higman [7]. More illuminatingly, denoting the n -th term of the lower central series of F by $\lambda_n F$, there exists an integer n such that $f \notin \lambda_n F$ (Magnus [9]). Moreover $F/\lambda_n F$ is torsion-free (Magnus [9]). Now the subgroups of prime index in a finitely generated torsion-free nilpotent group have trivial intersection (Higman

[7]). It follows that there is a nilpotent variety \underline{V} of prime exponent such that

$$f\lambda_n F \notin V(F/\lambda_n F) .$$

Hence

$$f \notin V(F)$$

as desired.

3. A crucial proposition

The main (indeed essentially the only) step in the proof of the theorem is the proof of the following proposition.

PROPOSITION. *Let G be a finitely generated group with a normal subgroup N such that G/N is infinite cyclic. If N is free in a nilpotent variety \underline{V} of prime exponent p then G is residually finite.*

Proof. We make use of Lemma 1 and Lemma 2 to describe this extension G of N by an infinite cyclic group in sufficiently concise terms so as to be able to deduce the residual finiteness of G .

To this end we begin by choosing $t \in G$ so that

$$G = \text{gp}(N, t) .$$

Since G/N is infinite cyclic, G is a split extension of N by $\text{gp}(t)$:

$$G = N\text{gp}(t) \quad \text{and} \quad N \cap \text{gp}(t) = 1 .$$

We put

$$M = N/N' .$$

Let us denote the group algebra of $\text{gp}(t)$ over the field of p elements by R . Then, writing M additively, we have $px = 0$ for every $x \in M$. So M may be regarded as an R -module once the action of t on M is defined (by conjugation):

$$aN' \cdot t = t^{-1}atN' \quad (a \in N) .$$

Now G is finitely generated. It follows that the R -module M is finitely generated. Hence, by Lemma 1, M is the direct sum of a finite number of cyclic modules:

$$M = M_1 \oplus \dots \oplus M_k \oplus M_{k+1} \oplus \dots \oplus M_l \quad (l \geq k) .$$

Our notation here has been chosen in such a way that M_1, \dots, M_k are free whereas M_{k+1}, \dots, M_l are all torsion-modules.

Let ε_i be a generator of M_i for each $i = 1, 2, \dots, l$. Furthermore let us denote the set of all integers by Z . Now put

$$\varepsilon_{i,j} = \varepsilon_i \cdot t^j \quad (j \in Z, i = 1, 2, \dots, k) .$$

Since M_i is free ($i = 1, 2, \dots, k$) it follows that the elements

$$\dots, \varepsilon_{i,-1}, \varepsilon_{i,0}, \varepsilon_{i,1}, \dots$$

are a basis for M_i , where here we regard M_i as a vector space over the field of p elements.

Consider now the submodules M_{k+1}, \dots, M_l . We choose positive integers n_{k+1}, \dots, n_l so that the elements

$$\varepsilon_{i,0}, \varepsilon_{i,1}, \dots, \varepsilon_{i,n_i-1} \quad (i = k+1, \dots, l)$$

constitute a basis for the vector space M_i where here

$$\varepsilon_{i,j} = \varepsilon_i \cdot t^j .$$

It follows that, for $i = k+1, \dots, l$,

$$(1) \quad \varepsilon_{i,n_i-1} \cdot t = m_{i,0} \varepsilon_{i,0} + \dots + m_{i,n_i-1} \varepsilon_{i,n_i-1}$$

where here $0 < m_{i,0} < p$.

This information can be re-expressed directly in terms of N and t . To this end let us choose $e_i \in N$ so that

$$\varepsilon_i = e_i N^i .$$

Now, putting

$$e_{i,j} = t^{-j} e_i t^j ,$$

it follows from the comments above that the following statements hold.

(i) The elements

$$(2) \quad \dots, e_{i,-1}, e_{i,0}, e_{i,1}, \dots$$

are linearly independent modulo N' for $i = 1, 2, \dots, k$, and similarly so too are the elements

$$(3) \quad e_{i,0}, e_{i,1}, \dots, e_{i,n_i-1}$$

for $i = k+1, \dots, l$.

(ii) The set E of all elements given by (2) and (3) generate N modulo N' and hence, remembering N is nilpotent, these elements actually generate N itself.

(iii) Since E is comprised of elements which are linearly independent modulo N' , E freely generates N (Lemma 2).

(iv) For each i satisfying $k+1 \leq i \leq l$ we have

$$(4) \quad t^{-1} e_{i,n_i-1} t = e_{i,0}^{m_{i,0}} \dots e_{i,n_i-1}^{m_{i,n_i-1}} f_i \quad (f_i \in N')$$

where the $m_{i,j}$ are those given by (1).

This information given by (i), (ii), (iii) and (iv) is sufficient for us to be able to deduce that G is residually finite. Thus suppose $g \in G$ ($g \neq 1$). Our objective is to find a homomorphism ϕ of G into a finite group such that $g\phi \neq 1$. If $g \notin N$ the existence of such a homomorphism is easily verified. Thus for the remainder of the proof of the proposition we shall assume that $g \in N$. We shall choose a homomorphic image \tilde{G} of G so that there is a homomorphism of G to \tilde{G} of the desired kind.

We repeat that $g \in N$. Since E generates N (see (ii)) there exists a positive integer n such that

$$\tilde{N} = \mathbb{E}P \left(e_{1,-n}, \dots, e_{1,0}, \dots, e_{1,n}; \dots; e_{k,-n}, \dots, e_{k,0}, \dots, e_{k,n}; \right. \\ \left. e_{k+1,0}, \dots, e_{k+1,n_{k+1}-1}; \dots; e_{l,0}, \dots, e_{l,n_l-1} \right)$$

contains all the elements

$$g, f_{k+1}, \dots, f_l.$$

It follows from (iii) that \tilde{N} is a free group in \underline{V} freely generated by the elements exhibited.

Our next move is to define an automorphism τ of \tilde{N} which mimics the action of t on N . The effect of τ on the generators of \tilde{N} is defined by

$$\begin{aligned} e_{1,-n}^\tau &= e_{1,-n+1}, \dots, e_{1,0}^\tau = e_{1,1}, \dots, e_{1,n}^\tau = e_{1,-n} \\ &\vdots \\ &\vdots \\ e_{k,-n}^\tau &= e_{k,-n+1}, \dots, e_{k,0}^\tau = e_{k,1}, \dots, e_{k,n}^\tau = e_{k,-n} \\ &\vdots \\ e_{k+1,0}^\tau &= e_{k+1,1}, \dots, e_{k+1,n_{k+1}-1}^\tau = e_{k+1,0}^{m_{k+1,0}} \dots e_{k+1,n_{k+1}-1}^{m_{k+1,n_{k+1}-1}} f_{k+1} \\ &\vdots \\ &\vdots \\ e_{l,0}^\tau &= e_{l,1}, \dots, e_{l,n_l-1}^\tau = e_{l,0}^{m_{l,0}} \dots e_{l,n_l-1}^{m_{l,n_l-1}} f_l. \end{aligned}$$

To see that τ does indeed define an automorphism, observe that the images of the given free generators of \tilde{N} generate \tilde{N} modulo \tilde{N}' . Hence they generate \tilde{N} since \tilde{N} is nilpotent. But \tilde{N} is finite. So τ is an automorphism of \tilde{N} (of finite order). Let τ be of order r and let $\text{gp}(\tilde{t})$ be a cyclic group of order r generated by \tilde{t} . Finally let \tilde{G} be the split extension of \tilde{N} by $\text{gp}(\tilde{t})$ with \tilde{t} inducing the automorphism τ of \tilde{N} :

$$\tilde{G} = \text{gp}(\tilde{N}, \tilde{t}; \tilde{t}^{-1}u\tilde{t} = u\tau \quad (u \in \tilde{N})).$$

The group \tilde{G} is clearly finite.

There is a natural homomorphism ϕ of G onto \tilde{G} defined by

$$\phi : t \rightarrow \tilde{t}, \quad e_{i,0} \rightarrow e_{i,0} \quad (1 \leq i \leq l).$$

To see that this mapping does define a homomorphism of G onto \tilde{G} it is enough to observe that the relations in G between the elements $e_{i,0}$ and

t are satisfied by the elements $e_{i,0}$ and \bar{z} - this follows from (2), (3) and (4). Clearly

$$g\varphi = g .$$

Hence $g\varphi \neq 1$ and the proof of the proposition is now complete.

4. Some final remarks

The proof of the theorem is now an immediate consequence of Lemma 3 and the Proposition once one observes that a finite extension of a residually finite group is residually finite.

It is worth pointing out that a finitely generated cyclic extension of a residually finite group need not be residually finite. Indeed if G is the wreath product of a free group U of rank two by an infinite cyclic group then G is not residually finite (Gruenberg [6], Theorem 3.2). But G is a finitely generated cyclic extension of a direct product F of free groups. Of course F is residually finite, but as we remarked G is not.

References

- [1] Gilbert Baumslag, "On generalised free products", *Math. Z.* 78 (1962), 423-438.
- [2] Gilbert Baumslag, "On the residual finiteness of generalised free products of nilpotent groups", *Trans. Amer. Math. Soc.* 106 (1963), 193-209.
- [3] Gilbert Baumslag, "Some subgroup theorems for free \underline{v} -groups", *Trans. Amer. Math. Soc.* 108 (1963), 516-525.
- [4] Bruce Chandler, "A representation of a generalized free product in an associative ring", *Comm. Pure Appl. Maths.* 21 (1968), 271-288.
- [5] Karen N. Frederick, "The Hopfian property for a class of fundamental groups", *Comm. Pure Appl. Maths.* 16 (1963), 1-8.
- [6] K.W. Gruenberg, "Residual properties of infinite soluble groups", *Proc. London Math. Soc.* (3) 7 (1957), 29-62.

- [7] Graham Higman, "A remark on finitely generated nilpotent groups",
Proc. Amer. Math. Soc. 6 (1955), 284-285.
- [8] Nathan Jacobson, *Lectures in abstract algebra*, Vol. II (Van Nostrand,
New York, Toronto, London, 1953).
- [9] Wilhelm Magnus, "Beziehungen zwischen Gruppen und Idealen in einem
speziellen Ring", *Math. Ann.* 111 (1935), 259-280.

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