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Finitely presented extensions by free groups

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Abstract. We prove here that a finitely presented group with a free quotient of rank *n* is an HNN-extension with *n* stable letters of a finitely generated group where the associated subgroups are finitely generated. This theorem has a number of consequences. In particular, in the event that the free quotient is cyclic it reduces to an elementary and quick proof of a theorem of Bieri and Strebel.

1 Finitely presented groups with free quotients

Our main objective in this note is to prove the following

Theorem 1. *Let G be a finitely presented group with a free quotient of rank n. Then G is an HNN extension with n stable letters, of a finitely generated group B and finitely generated associated subgroups.*

It is worth noting that if N is the normal closure in G of B, then N is the fundamental group $\pi(\Gamma)$ of a graph Γ of groups, where Γ is the Cayley graph of a free group of rank n. The vertex groups are isomorphic to B and the edge groups are isomorphic to the associated subgroups involved in the decomposition of G as an HNN-extension and are therefore all finitely generated.

There are a number of consequences of Theorem 1. The first of these is the following theorem of Bieri and Strebel [3] which is a special case of Theorem 1.

Corollary 1. *A finitely presented group with an infinite cyclic quotient is an HNN extension of a finitely generated group with a single stable letter and finitely generated associated subgroups.*

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In order to formulate the second consequence of Theorem 1, we recall that a group is termed *coherent* if all of its finitely generated subgroups are finitely related. We then have another immediate consequence of Theorem 1.

Corollary 2. *Let N be a normal, coherent subgroup of a finitely presented group G. If G/N is a free group of rank n, then G is an HNN extension of a finitely presented group with n stable letters and finitely presented associated subgroups. In particular, if G is a finitely presented group which is an extension of a locally free group by a free group of rank n, then G is an HNN extension of a finitely generated free group with n stable letters and finitely generated free associated subgroups.*

Now suppose that G is an HNN extension of the group B. If for every pair of associated subgroups at least one of these subgroups coincides with B, we term G an *ascending* HNN extension. For instance the metabelian group $G = \langle a, t \mid t^{-1}at = a^2 \rangle$ is an ascending HNN extension. On the other hand non-ascending HNN extensions always have non-abelian free subgroups. More precisely, if G is an HNN extension of the base group B which not an ascending HNN extension with respect to one of its stable letters, then the normal closure N of B contains a free subgroup of rank 2 (see Lemma 7 below for a very elementary proof). This fact allows us to draw some extra consequences from Theorem 1 under some additional assumptions

Theorem 2. *Suppose that $\mathscr{P}$ is a property of groups closed under subgroups and suppose that non-abelian free groups do not have $\mathscr{P}$. Let N be a normal subgroup of the finitely presented group G. If the finitely generated subgroups of N belong to $\mathscr{P}$ and if G/N is free, then G is an ascending HNN extension of a finitely generated group $B \in \mathscr{P}$ with finitely generated associated subgroups. If N is coherent, then B and the associated subgroups are also finitely presented.*

Now let $\mathscr{V}$ be a non-trivial variety of groups. The following corollary is then a special case of Theorem 2.

Corollary 3. *Let N be a normal subgroup of a finitely presented group G. If G/N is free and $N \in \mathscr{V}$, then G is an ascending HNN extension of a finitely generated group B in $\mathscr{V}$ with finitely generated associated subgroups.*

Here is another immediate consequence of Theorem 2.

Corollary 4. *Suppose that N is a normal, locally polycyclic subgroup of the finitely presented group G. If G/N is free, then G is an ascending HNN extension of a polycyclic subgroup B.*

It follows that if N is an abelian normal subgroup of the finitely presented group G and if G/N is free, then G is an ascending HNN extension of a finitely generated abelian subgroup B of N.

Finally we note the following further consequence.

Corollary 5. *Suppose that N is a normal subgroup of the finitely presented group G. If N is torsion and G/N is free, then G is an ascending HNN extension of a finitely generated torsion subgroup B of N with finitely generated associated subgroups. In particular, if N is locally finite, then G is an ascending HNN extension of a finite subgroup, and hence N is finite and G is virtually free.*

2 The proof of Theorem 1

Proof. By hypothesis G has a quotient group G/N which is free of rank n. Since G is finitely generated we can choose a finite set of generators

$$t_1, \dots, t_n, \quad a_1, \dots, a_m$$

of G where $t_1N, \dots, t_nN$ freely generate G/N and $a_1, \dots, a_m \in N$. Since G is finitely presented, it can be presented on these generators and defined in terms of them by a finite number of relations as

$$G = \langle t_1, \dots, t_n, a_1, \dots, a_m \,|\, r_1 = 1, \dots, r_k = 1 \rangle.$$

Each of the relators r_j is a word in the given generators and so takes the form

$$r_j = u_1 v_1 u_2 \dots u_l v_l$$

where the u_i are words in the generators $t_1, \dots, t_n$ and the v_i are words in the generators $a_1, \dots, a_m$. Since $r_j = 1$ in G we have $r_j N = u_1 \dots u_l N = N$. Now the images of the t_i are a free basis for G/N, so that $u_1 u_2 \dots u_l = 1$ (freely) and thus $u_l = u_{l-1}^{-1} \dots u_1^{-1}$. Hence we can rewrite r_j as follows:

$$r_j = (u_1 v_1 u_1^{-1})(u_1 u_2 v_2 u_2^{-1} u_1^{-1}) \dots (u_1 u_2 \dots u_{l-1} v_{l-1} u_{l-1}^{-1} \dots u_1^{-1} v_l).$$

This means that each r_j can be expressed as a product of conjugates of the a_h by various words in the t_i. Let δ be the maximum of the lengths of these words in t_i that arise in such expressions, and let Ω be the set of all freely reduced words in the t_i of length at most δ.

We now Tietze transform the presentation for G by introducing the new generators $\beta_{h,u}$, where u ranges over Ω and $h = 1, \dots, m$, and the following relations defining them in terms of the existing generators:

$$u^{-1} a_h u = \beta_{h,u} \quad (u \in \Omega, h = 1, \dots, m).$$

Notice that in particular $a_h = \beta_{h,1}$ $(h = 1, \dots, m)$. So we have the new presentation

$$G = \langle t_1, \dots, t_n, a_1, \dots, a_m, \beta_{h,u} \ (u \in \Omega, h = 1, \dots, m) \,|\, r_1 = 1, \\ \dots, r_k = 1, u^{-1} a_h u = \beta_{h,u} \ (u \in \Omega, h = 1, \dots, m) \rangle.$$

As a consequence of the relations $u^{-1}a_h u = \beta_{h,u}$ our earlier observations imply each of the words r_j is equal to a word w_j in the $\beta_{h,u}$. So we can replace the r_j by the w_j in our presentation for G to get

$$G = \langle t_1, \dots, t_n, a_1, \dots, a_m, \beta_{h,u}\ (u \in \Omega, h = 1, \dots, m) \,|\, w_1 = 1, \\ \dots, w_k = 1, u^{-1}a_h u = \beta_{h,u}\ (u \in \Omega, h = 1, \dots, m)\rangle.$$

Using the equations $a_h = \beta_{h,1}$ we can now eliminate the generators a_h and obtain the presentation

$$G = \langle t_1, \dots, t_n, \beta_{h,u}\ (u \in \Omega, h = 1, \dots, m) \,|\, w_1 = 1, \dots, w_k = 1, \\ u^{-1}\beta_{h,1}u = \beta_{h,u}\ (u \in \Omega, h = 1, \dots, m)\rangle.$$

Next we observe that the set of relations

$$u^{-1}\beta_{h,1}u = \beta_{h,u} \quad (u \in \Omega, h = 1, \dots, m)$$

is equivalent to the set of relations

$$t_i^{-1}\beta_{h,u}t_i = \beta_{h,ut_i} \quad (u \in \Omega, ut_i \in \Omega, 1 \leqslant i \leqslant n, 1 \leqslant h \leqslant m).$$

So applying further Tietze transformations we can present G as

$$G = \langle t_1, \dots, t_n, \beta_{h,u}\ (u \in \Omega, h = 1, \dots, m) \,|\, w_1 = 1, \dots, w_k = 1, \\ t_i^{-1}\beta_{h,u}t_i = \beta_{h,ut_i}\ (u \in \Omega, ut_i \in \Omega, 1 \leqslant i \leqslant n, 1 \leqslant h \leqslant m)\rangle.$$

Now put

$$B = \mathrm{gp}(\beta_{h,u} \,|\, h = 1, \dots, m, u \in \Omega),$$

which is clearly finitely generated, but may not be finitely presented. A system of defining relations for B in terms of these generators are consequences of the defining relations for G. We can therefore add these relations (none of which involve the elements t_j) to the ones above and the resultant presentation is again a presentation for G.

Now the subgroup C_i of B generated by the elements

$$\{\beta_{j,u} \,|\, 1 \leqslant j \leqslant k, u \in \Omega \text{ and } ut_i \in \Omega\}$$

is conjugate in G by t_i (and hence isomorphic) to the subgroup D_i generated by

$$\{\beta_{j,ut_i} \,|\, 1 \leqslant j \leqslant k, u \in \Omega \text{ and } ut_i \in \Omega\}.$$

Thus we have exhibited G as an HNN extension of the finitely generated group B. $\square$

3 Non-ascending HNNs have free subgroups

Recall that an HNN extension $G = \langle B, t \mid t^{-1}Ct = D \rangle$ is said to be *ascending* if either $C = B$ or $D = B$. Let N be the normal closure of B in the HNN extension G. The structure of N is very different in the ascending and non-ascending cases.

If G is ascending with say $C = B$ then we have

$$\ldots \supseteq t^2Bt^{-2} \supseteq tBt^{-1} \supseteq B \supseteq t^{-1}Bt \supseteq \ldots$$

so that $N = \bigcup_{n \in \mathbb{Z}} t^{-n}Bt^n$. Thus N is the union of copies of B.

On the other hand if G is not ascending so that $C \neq B \neq D$ then N has the structure of a two-way infinite amalgamated free product

$$\ldots tBt^{-1} *_{tDt^{-1}=C} B *_{D=t^{-1}Ct} t^{-1}Bt *_{t^{-1}Dt=t^{-2}Ct^2} t^{-2}Bt^2 \ldots$$

where the amalgamations are proper (see for example [4] or [5]). From this one can easily deduce that N has non-abelian free subgroups. Alternatively, an elementary proof of this folklore fact can be given by considering two elements $u = t^{-1}h_0th_1$ and $v = th_1t^{-1}h_0$ where $h_0 \in B \backslash C$ and $h_1 \in B \backslash D$. An examination of cases shows that a freely reduced word in u and v is necessarily t-reduced. By Britton's Lemma such a word is not equal to 1 in G, and hence u and v freely generate a free subgroup of N.

Our discussion can be summarized as follows.

Lemma 6. *Suppose that $G = \langle B, t \mid t^{-1}Ct = D \rangle$ is an HNN extension and let N be the normal closure of B in G.*

(1) *If G is ascending, then N is is a union of subgroups isomorphic to B.*

(2) *If G is not ascending, then N contains non-abelian free subgroups.* $\square$

There is a straightforward generalization of this to HNN extensions with several stable letters. The HNN extension

$$G = \langle B, t_1, t_2, \ldots \mid t_1^{-1}C_1t_1 = D_1, t_2^{-1}C_2t_2 = D_2, \ldots \rangle$$

is said to be *ascending* if for each $i = 1, 2, \ldots$ either $C_i = B$ or $D_i = B$. In the case of several stable letters, there does not seem to be a nice description of N as an ascending union. However, in the non-ascending case the argument given above applies to show the existence of non-abelian free subgroups using a non-ascending stable letter. We record this as follows.

Lemma 7. *Suppose that*

$$G = \langle B, t_1, t_2, \dots \mid t_1^{-1} C_1 t_1 = D_1, t_2^{-1} C_2 t_2 = D_2, \dots \rangle$$

is an HNN extension and let N be the normal closure of B in G. If G is not ascending, then N contains non-abelian free subgroups. $\square$

4 Some examples

We indicate here how Theorem 1 can be used to decompose various finitely presented groups as HNN extensions. The first example is well known, illustrating that the fundamental group

$$G_k = \langle a_1, b_1, \dots, a_k, b_k; [a_1, b_1] \dots [a_k, b_k] = 1 \rangle$$

of an orientable surface of genus at least 1 can be expressed as an HNN extension with a single stable letter, finitely generated free base group and finitely generated associated subgroups.

Example 1. Let N be the normal closure in G_k of $a_2, \dots, a_k, b_1, \dots, b_k$. Then N is free and G/N is infinite cyclic. So G_k must be an HNN extension of a free group. Indeed, G_k is an HNN extension of the free group on $a_2, \dots, a_k$, $b_1, \dots, b_k$, stable letter a_1 and associated subgroups $\mathrm{gp}(b_1)$ and $\mathrm{gp}(b_1[a_2, b_2] \dots [a_k, b_k])$.

Example 2. Let N be the normal closure in G_k of $b_1, \dots, b_k$. Since every subgroup of infinite index in G_k is free, N is free and G/N is free of rank k. So G must be an HNN extension with k stable letters of a free group. Indeed G_k is an HNN extension of the free group B on

$$b_1, b_2, b_2^{a_2}, \dots, b_k, b_k^{a_k}$$

with stable letters $a_1, \dots, a_k$. The associated subgroups for a_1 can be chosen to be $\mathrm{gp}(b_1)$ and $\mathrm{gp}(b_1 (b_2^{-1})^{a_2} b_2 \dots (b_k^{-1})^{a_k} b_k)$ and those for the a_j, where $j > 1$, can be chosen to be $\mathrm{gp}(b_j)$ and $\mathrm{gp}(b_j^{a_j})$.

Example 3. Let

$$G = \langle a, s, t; a^t = a a^s, [a, a^s] = 1, [s, t] = 1 \rangle.$$

Since G maps onto the infinite cyclic group on t, G is an HNN extension of a finitely generated group. Now G turns out to be metabelian [2]. Consequently it must be an ascending HNN extension. Indeed G is an ascending HNN extension of the wreath product of two infinite cyclic groups with a single stable letter. Notice that in this case the base group is not finitely presented [1].

Example 4. Let

$$G = \langle a, s, t; a^t = a a^s, [a, a^s] = 1 \rangle.$$

Notice that G maps onto the free group of rank 2 on s and t. So G is an HNN extension with two stable letters of a finitely generated group B and finitely generated associated subgroups. Here B turns out to be the free abelian group on a and a^s. The stable letters are s and t. The associated subgroups of s are $\mathrm{gp}(a)$ and $\mathrm{gp}(a^s)$, while those of t are $\mathrm{gp}(a)$ and $\mathrm{gp}(aa^s)$. It follows, using Bass–Serre theory, that the finitely generated subgroups of G are finitely presented since the subgroups of the vertex and edge groups are either trivial, cyclic or free abelian of rank 2.

References

[1] G. Baumslag. Wreath products and finitely presented groups. *Math. Z.* **75** (1961), 22–28.
[2] G. Baumslag. A finitely presented metabelian group with a free abelian derived group of infinite rank. *Proc. Amer. Math. Soc.* **35** (1972), 61–62.
[3] R. Bieri and R. Strebel. Almost finitely presented soluble groups. *Comment. Math. Helv.* **53** (1978), 258–278.
[4] G. Higman. Subgroups of finitely presented groups. *Proc. Royal Soc. London Ser. A* **262** (1961), 455–475.
[5] A. Karrass and D. Solitar. Subgroups of HNN groups and groups with one defining relation. *Canad. J. Math.* **23** (1971), 627–643.
[6] R. C. Lyndon and P. E. Schupp. *Combinatorial group theory* (Springer-Verlag, 1977).

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