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Free subgroups of certain one-relator groups defined by positive words

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1. *Introduction*

1·1. Let $\mathscr{Z}$ be the class of those groups G which can be presented in the form

$$G = \langle a, b, \ldots, c;\ uv^{-1} = 1\rangle,$$

where u and v are positive words in the given generators. Here a word w is termed *positive* if only non-negative powers of $a, b, \ldots, c$ occur in w. If each generator occurs with exponent sum zero in uv^{-1}, we term the $\mathscr{Z}$-group G a $\mathscr{Y}$-group. This class $\mathscr{Y}$ contains, in particular, the class $\mathscr{X}$ of those groups G which can be presented in the form

$$G = \langle a, b, \ldots, c;\ [u, v] = 1\rangle,$$

where u and v are positive words, and where $[u, v]$ is the commutator $uvu^{-1}v^{-1}$.

Clearly $\mathscr{X} \subseteq \mathscr{Y} \subseteq \mathscr{Z}$. It is obvious that $\mathscr{Y} \neq \mathscr{Z}$ and, although not quite so obvious, easy to show that $\mathscr{X} \neq \mathscr{Y}$ (see (**4**)).

1·2. THEOREM 1. *Every $\mathscr{Y}$-group is free-by-cyclic. More precisely every $\mathscr{Y}$-group can be embedded in a two generator $\mathscr{Y}$-group, and every two-generator $\mathscr{Y}$-group is an extension of a finitely generated free group by a cyclic group.*

Since cyclic extensions of finitely generated residually finite groups are again residually finite (**10**), it follows that

COROLLARY 1. *$\mathscr{Y}$-groups are residually finite.*

Even $\mathscr{X}$-groups can be quite complicated. For example, consider the group $G = \langle a, b, c;\ a^2b^2c^2b^{-2}a^{-2}c^{-2} = 1\rangle$. Clearly G is an $\mathscr{X}$-group. Moreover G contains the group $H = \langle x_1, x_2, x_3, x_4;\ x_1^2x_2^2x_3^2x_4^2 = 1\rangle$; in fact we can take (cf. (**4**)) $x_1 = a$, $x_2 = b$, $x_3 = c^2b^{-1}c^{-2}$, $x_4 = c^2a^{-1}c^{-2}$. It follows that G contains the fundamental group of every compact two-dimensional orientable manifold (cf. e.g. (**4**)). Thus Corollary 1 yields an easy proof of the residual finiteness of compact surface groups. This has been proved in a number of other ways (see e.g. (**1**), (**2**) and (**5**)).

1·3. There exist one-relator groups of the form $G = \langle a, b, \ldots, c;\ [u, v] = 1\rangle$ which are not free-by-cyclic (see § 4). Nevertheless we have

THEOREM 2. *Let $G = \langle a, b, \ldots, c;\ [u, v] = 1$, where no generator appears in both u and v. Then* (i) *some term of the lower central series of G is free;* (ii) *G is residually finite.*

The rest of this note is arranged as follows. Theorem 1 is proved in § 2, Theorems 2 and 3 are dealt with in §§ 3 and 4 and § 4 contains some examples, ending with a slight variant of Theorem 1.

2. *The proof of Theorem* 1

2·1. In order to prepare for the proof of Theorem 1 we need to recall some notation and an elementary result about one-relator groups.

Suppose then that G is a group, X a subset of G. Then we respectively denote the subgroup and the normal subgroup of G generated by X by $gp(X)$ and $gp_G(X)$. If G is generated by $b, x, \ldots, c$ we put

$$x_i = b^i x b^{-i}, \ldots, c_i = b^i c b^{-i} \quad (i \in \mathbb{Z}).$$

Notice that if $w = w(b, x, \ldots, c)$ is a word in the given generators of G in which b occurs with exponent sum zero, then w can be rewritten as a word in $\ldots, x_i, \ldots, c_i, \ldots$:

$$w = w_0(\ldots, x_i, \ldots, c_i, \ldots).$$

We then define w_k to be the word obtained from w_0 by adding k to each of the subscripts of each of the generators occurring in w_0:

$$w_k = w_0(\ldots, x_{i+k}, \ldots, c_{i+k}, \ldots) \quad (k \in \mathbb{Z}).$$

The following lemma is an immediate consequence of W. Magnus' basic breakdown of groups with a single defining relation (9):

LEMMA 1. *Let*

$$G = \langle b, x, \ldots, c; r = 1 \rangle$$

be a group with a single defining relation. Suppose that b occurs in r with exponent sum zero and that μ and ν are respectively the minimum and maximum subscripts of x occurring in r_0. If $\mu < \nu$ and if both x_μ and x_ν occur only once in r_0 then

$$N = gp_G(x, \ldots, c)$$

is free. Moreover if G is a two-generator group with generators b and x, then G is free of rank $\nu - \mu + 1$.

2·2. There are two steps in the proof of Theorem 1. The first is to prove that every $\mathscr{Y}$-group can be embedded in a two-generator $\mathscr{Y}$-group. This allows us to restrict attention to such two-generator $\mathscr{Y}$-groups $G = \langle a, b; uv^{-1} = 1 \rangle$.

The second step in the proof is then, after a simple 'change of variable' $a = xb$, to verify that $N = gp_G(x)$ is free. Thus the proof of Theorem 1 turns out to be an exercise in keeping track of subscripts that are generated in a rather simple way.

We begin with

LEMMA 2. *Every $\mathscr{Y}$-group can be embedded in a two-generator $\mathscr{Y}$-group.*

Proof. Suppose that G is presented in the form $G = \langle a, b, \ldots, c, d; uv^{-1} = 1 \rangle$, where u and v are positive words. We may assume without loss of generality that uv^{-1} is a cyclically reduced word that involves all of the generators. It follows from the Freiheitssatz (W. Magnus (8)) that both $H = gp(a, b, \ldots, c)$ and $K = gp(b, c, \ldots, d)$ are free on the generators displayed (of the same rank). Thus G embeds in the HNN-extension G^+ (see e.g. R. C. Lyndon and P. E. Schupp (7)):

$$G^+ = \langle G, t; t^{-1}at = b, \ldots, t^{-1}ct = d \rangle.$$

This construction of G^+ is, of course, well known. Clearly G^+ is generated by a and t. To see that G^+ is a $\mathscr{Y}$-group, we put $a = t^{n-1}s$. Then $b = t^{n-2}st, \ldots, c = tst^{n-2}$, $d = st^{n-1}$. So if $u = u(a, b, \ldots, c, d)$ and $v = v(a, b, \ldots, c, d)$ then u and v can both be expressed as positive words in s and t. This completes the proof of the lemma.

2·3. We turn now to the second step in the proof of Theorem 1. Thus suppose $G = \langle a, b;\ uv^{-1} = 1\rangle$ is a $\mathscr{Y}$-group. There are two cases to consider.

(i) $u = a^{\alpha_1}b^{\beta_1}\ldots a^{\alpha_m}b^{\beta_m}, \quad v = b^{\gamma_1}a^{\delta_1}\ldots b^{\gamma_n}a^{\delta_n} \quad (\alpha_i > 0, \beta_j > 0, \gamma_k > 0, \delta_l > 0)$;

(ii) $u = a^{\alpha_1}b^{\beta_1}\ldots a^{\alpha_m}b^{\beta_m}a^{\alpha_{m+1}}, \quad v = b^{\gamma_1}a^{\delta_1}\ldots b^{\gamma_n}a^{\delta_n}b^{\gamma_{n+1}}$

$$(m \geqslant 0, n \geqslant 0, \alpha_i > 0, \beta_j > 0, \gamma_k > 0, \delta_l > 0).$$

Notice that in both cases $\Sigma\alpha_i = \Sigma\delta_i$ and $\Sigma\beta_i = \Sigma\gamma_i$.

Now put $a = xb$. Then b occurs with exponent sum zero in uv^{-1}. So we can express $w = uv^{-1}$ in the form $w = w_0(x_\mu, \ldots, x_\nu)$, where μ and ν are respectively the minimum and maximum subscripts occurring. An easy calculation reveals, in case (i), that $\mu = 0$ and $\nu = \gamma_1 + \ldots + \gamma_n + \delta_1 + \ldots + \delta_n - 1$; and that these subscripts μ and ν occur exactly once. Similarly we find, in case (ii), that

$$\mu = 0 \quad \text{and} \quad \nu = \alpha_1 + \ldots + \alpha_{m+1} - 1 + \beta_1 + \ldots + \beta_m;$$

and again these subscripts μ and ν occur exactly once. It follows then from Lemma 1 that G is a cyclic extension of a finitely generated free group. Theorem 1 follows immediately on appealing to Lemma 2.

3. *The proofs of Theorems* 2 *and* 3

3·1. We prove first Theorem 2: if $G = \langle a, b, \ldots, c;\ [u, v] = 1\rangle$ and if no generator appears in both u and v, then some term of the lower central series of G is free.

The key fact needed is the following lemma, which can be deduced from the main theorem of A. Karrass and D. Solitar (**6**).

LEMMA 3. *Let* G_1, G_2, *and* A *be given groups and suppose that*

$$G_1 \cap A = H, \quad G_2 \cap A = K.$$

Furthermore let

$$F = \{G_1 * A;\ H\}, \quad G = \{F * G_2;\ K\}$$

be generalized free products with amalgamated subgroups H and K respectively. Finally let M be a normal subgroup of G which meets A trivially. Then M is a free product of conjugates of subgroups of G_1 *and* G_2 *and a free group.*

It suffices for the proof of Lemma 3 to note that the desired conclusion follows from the cited theorem of Karrass and Solitar on applying it first to G and then to F.

We are now in a position to prove Theorem 2. To this end we relabel the generators $x_1, \ldots, y_1, \ldots$ of G in such a way as to ensure that u is a word in $x_1, \ldots$ and v is a word in $y_1, \ldots$. G is clearly free if either u or v is trivial. So we may assume that $u \neq 1, v \neq 1$. Let G_1 be the free group on $x_1, \ldots$, G_2 the free group on $y_1, \ldots$ and A the free abelian group on x and y. It follows immediately (W. Magnus, A. Karrass and D. Solitar (**12**), p. 220, Exercises 22 and 23) that if F is the generalized free product $F = \{G_1 * A;\ u = x\}$, then $G = \{F * G_2;\ v = y\}$.

According to W. Magnus (10) free groups are residually nilpotent. Therefore we can find normal subgroups M_1 and M_2 of G_1 and G_2 respectively with nilpotent factor groups G_1/M_1, G_2/M_2 such that $u \notin M_1$, $v \notin M_2$.

Put $D = G_1/M_1 \times G_2/M_2$. Then D is nilpotent.

Let now ϕ_1, α and ϕ_2 be respectively the homomorphisms of G_1, A and G_2 into D defined by $\phi_i\colon g_i \mapsto g_i M_i$ $(g_i \in G_i)$, $i = 1, 2$, $\alpha\colon x \mapsto uM_1$, $y \mapsto vM_2$. Then ϕ_1, α and ϕ_2 can be extended to a homomorphism $\phi\colon G \mapsto D$. Since ϕ is injective on A the kernel M of ϕ meets A trivially. Hence M is, by Lemma 3, a free product of conjugates of subgroups of the free groups G_1 and G_2 and a free group. Thus M is free. But G/M is nilpotent; so the proof of Theorem 2 is complete.

3·2. The proof that the groups G in Theorem 2 are residually finite is an easy application of the methods introduced in (2). Indeed if one keeps track of the way in which G has been constructed in 3·1, it turns out that Proposition 2 of G. Baumslag (2) applies. The details are left to the reader.

4. *Some examples and further comments*

4.1. The classes $\mathscr{Y}$ and $\mathscr{X}$ are distinct. Indeed the group $G = \langle a, b; (ab)^3 = ba^2b^2a\rangle$ is clearly a $\mathscr{Y}$-group. But it follows from an algorithm of M. Wicks (13) that $(ab)^3(ba^2b^2a)^{-1}$ is not a commutator, so G is not in $\mathscr{X}$.

4·2. Now all positive one-relator group are residually solvable (G. Baumslag (3)). However this is not so for $\mathscr{X}$-groups. Indeed let $G = \langle a, b;\ a = [a, a^b]\rangle$. If we expand the defining relation for G it takes the form $a = a^{-1}b^{-1}a^{-1}bab^{-1}ab$ or $aba^2 = bab^{-1}ab$. On putting $a = bx$ this relation becomes $bxb^2xbx = bx^2b$ or, more simply, $b^2xbx = xb$. So G is in $\mathscr{X}$. However bx lies in every term of the derived series of G and as $bx \neq 1$, G is not residually solvable.

4·3. If u and v are positive words in the generators $a, b, \ldots, c$ then

$$G = \langle a, b, \ldots, c;\ [u, v] = 1\rangle$$

is a $\mathscr{Y}$-group. Hence G is free-by-cyclic and therefore residually solvable. If we relax the condition that u and v be positive, then it turns out that G need no longer be residually solvable. Indeed put $u = a$ and $v = [a, b]\,[w, w^b]$, where $w = [a, b]^{-1}\,[a, b]^a$. Now it follows from Magnus' solution of the word problem in (9) that $w \neq 1$. Furthermore since $[u, v] = 1$ we find that $[a, b]^a\,[w, w^b]^a = [a, b]\,[w, w^b]$. Hence

$$w = [a, b]^{-1}\,[a, b]^a = [w, w^b]\,([w, w^b]^a)^{-1},$$

so w lies in every term of the derived series of G.

4.4. For our final example, take F to be the free group on a and b and let v be an element in the kth term of the derived series of F but not in the $(k+1)$*st*. Put $G = \langle a, b; [a, v^2] = 1\rangle$. Then $(a^{-1}va)^2 = v^2$ but $a^{-1}va \neq v$. So the kth term of the derived series of G is not free, since it contains unequal elements with equal squares. This example shows that Theorem 3 cannot be sharpened in the obvious way.

4·5. Now suppose that $G = \langle a, b;\ uv^{-1} = 1\rangle$ where u and v are positive, and that both generators appear in uv^{-1}. If α is the exponent sum of a in uv^{-1} and if β is that of b, then G is again free-by-cyclic if $\alpha > 0 > \beta$. This follows on evening out exponent sums and proceeding along the lines of the proof of Theorem 1. Just how far and in which other directions one can go is unclear. (cf. (**11**))

4·6. In conclusion I would like to thank both R. B. J. T. Allenby and S. J. Pride for pointing out that my original proof of Theorem 1, though essentially that given here, was much more complicated than necessary. I also acknowledge gratefully support from the N.S.F. and the S.R.C. and the hospitality of the University of Cambridge.

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