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On Some Finiteness Properties in Infinite Groups

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Abstract. We consider some questions concerning finiteness properties in infinite groups which are related to Marshall Hall's theorem. We call these properties Property S and Property R, and they are trivially true in finite groups. We give several elementary proofs using these properties for results on finitely generated subgroups of free groups as well as a new elementary proof of Hall's basic result. Finally, we consider these properties within surface groups and prove an analog of Hall's theorem in that context.

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1 Introduction

We consider a collection of questions concerning some finiteness properties in infinite groups which are related to Marshall Hall's theorem. If A and B are subgroups of a group G, then A and B are *commensurable* if $A \cap B$ has finite index in both A and B. A group G satisfies Property S if whenever A and B are finitely generated commensurable subgroups of G, then $A \cap B$ also has finite index in the join $\langle A, B\rangle$. From a theorem of Greenberg [13] and Stallings [40], free groups satisfy Property S. From a result of Mal'cev, finitely generated nilpotent groups also satisfy Property S. Further, a finitely generated group G satisfies Property R if each finitely generated subgroup H of G has the property that a subgroup of H is of finite index in H if and only if there exists a sufficiently large power of each generator in any finite generating system of this subgroup. Nilpotent groups satisfy Property R, and we show in general that if a group G satisfies Property R, then G also satisfies Property S. Further, a restricted version is true in torsion-free hyperbolic groups. In this paper, we give a new simple proof of the Greenberg–Stallings result based on Hall's theorem (see Section 2) that any finitely generated subgroup of a free group is virtually a free factor. We also give a new alternative proof of Hall's result based on the subgroup graph. Several other basic results on free groups then follow using the same technique.

A group G is a *Marshall Hall group* if any finitely generated subgroup H of G is virtually a free factor. Brunner and Burns [7] determined a structure theorem for Hall groups. Our proof of Property S for free groups can be extended to prove that any Hall group has a restricted version of Property S, where the subgroups A and B are infinite. Further, any virtually Hall group or more generally any virtually free group also satisfies this restricted version of Property S.

The Marshall Hall property was introduced originally to handle subgroup separability (see Section 3) in free groups. As an outgrowth of these investigations, we show that a large subclass of the set of hyperbolic limit groups are subgroup separable (see Section 4). Recall that a *limit group* is a finitely generated fully residually free group (see [5]).

Kapovich and Short [16] extended Property S to torsion-free hyperbolic groups with the restriction that the subgroups A and B must be quasi-convex. In particular, this shows that hyperbolic surface groups satisfy Property S. By a surface group we mean the fundamental group of a compact surface of finite genus. We interpret this in the following interesting way: if A and B are finitely generated commensurable free subgroups of a surface group, then their join $\langle A, B\rangle$ is a free group. We pose the question as to whether there is a purely algebraic proof of this result. In considering this question, we prove a Marshall Hall type theorem for surface groups. In another direction, Kapovich [25] showed that all fully residually free groups satisfy Property S. We give a separate proof of Kapovich's result in the case where they are hyperbolic — the hyperbolic limit groups.

The outline of this paper is as follows. In Section 2, we present the new elementary proofs of both Hall's theorem and the Greenberg–Stallings result. We then use the same technique to derive simple proofs of several additional well-known results on free groups. Many of the results in this section were also independently reproved in a different manner by Kapovich and Myasnikov [15] using Stallings foldings. The techniques we use are more traditional and we believe somewhat more elementary. In Section 3, we consider the situation in Hall groups and show that any virtually free group satisfies a restricted version of Property S. In Section 4, we consider the case of surface groups and prove a Marshall Hall type theorem. In Section 5, we introduce Property R and show that as a consequence of this property, Property S is true in finitely generated nilpotent groups and in certain finite extensions of such groups. In Section 6, we give a non-virtually nilpotent example of a group satisfying Property R and hence Property S. In the final section, we pose a collection of problems.

2 Marshall Hall's Theorem and Property S

In this section, we present a new elementary proof of Hall's theorem (Theorem 2.1) and then use this theorem to give a simple proof of Property S for free groups. Recall that if $\mathcal{P}$ is a group property, then a group G is *virtually* satisfies $\mathcal{P}$ if G has a subgroup of finite index satisfying $\mathcal{P}$.

Theorem 2.1. (Marshall Hall) *Let H be a finitely generated subgroup of a free group F. Then H is virtually a free factor, that is, there is a subgroup G of finite index in F for which H is a free factor.*

The original proof of Hall used Schreier transversals and a map into a finite permutation group (see [3] and [27]). This proof has been translated into geometric language and a proof based on graph theory has been given. These graph-theoretic ideas were extended to torsion-free hyperbolic groups and Theorem 2.1 can be reobtained via a result of Stallings, Kapovich and Short (see [16]). Below we give a simple direct proof.

Proof of Theorem 2.1. Without loss of generality we may assume that F is freely generated by a finite set X. Let (Γ_H, v) be a finite labeled graph with a distinguished vertex v representing H. This means that the following conditions are satisfied:

(1) The graph Γ_H is connected.

(2) The edges of this graph are labeled by letters from $X \cup X^{-1}$. If an edge e is labeled by x, the edge e^{-1} is labeled by x^{-1}.

(3) For any vertex of this graph, the edges going out of this vertex have different labels.

(4) H is generated by the labels of loops at v.

This graph can be obtained in the following way. Let $w_1, \dots, w_n$ be a Nielsen reduced set of generators for H. Draw a wedge of loops with a common vertex v. Divide each loop into edges and label them so that along the i-th edge we can read the word w_i. Then for any pair of these loops, we will identify their largest initial and terminal segments having the same labels.

For example, let $H = \langle ab^{-2}, ba^{-2} \rangle \leqslant F(a, b)$.

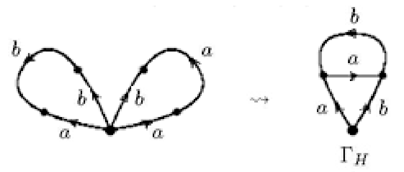

Figure 1

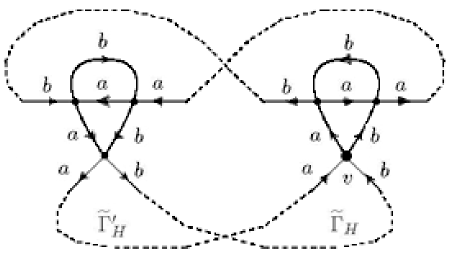

Figure 2

A vertex u of Γ_H will be called *full* if the number of edges going out of u is maximal possible, that is $2n$. If each vertex of a subgroup graph is full, then the subgroup must be of finite index. Now we add some edges to the graph Γ_H to make each of the vertices full. The resulting graph will be denoted by $\tilde{\Gamma}_H$. Let $\tilde{\Gamma}'_H$ be a copy of the graph $\tilde{\Gamma}_H$ with the inverse edge labeling, that is, if $\gamma : \tilde{\Gamma}_H \to \tilde{\Gamma}_H$ is the isomorphism, then the label of any edge e of $\tilde{\Gamma}_H$ is inverse to the label of $\gamma(e)$. Finally, we identify the outer edges of $\tilde{\Gamma}_H$ with the corresponding edges of $\tilde{\Gamma}'_h$. We now obtain the labeled graph Δ_H which has each edge full. This graph (with distinguished vertex v) represents a subgroup G of F. The subgroup H is a free factor of G since Γ_H is a subgraph of Δ_H. Moreover, G is a subgroup of finite index in F since all the vertices of Δ_H are full. □

We now use Hall's theorem to consider Property S in free groups (the Greenberg–Stallings theorem).

Definition 2.2. A group G has *Property* S if for any finitely generated commensurable subgroups A and B of G, $A \cap B$ has finite index in $\langle A, B \rangle$, the join in G of A and B.

Clearly, any finite group or locally finite group satisfies Property S, so in the infinite case, this property can be considered a fairly strong finiteness result. The genesis of this concept is the following theorem of Greenberg [13] and independently Stallings [40].

Theorem 2.3. *Free groups satisfy Property* S.

In order to prove this using Hall's theorem, we need the following classical concept which can also be found in [15]. The conjugate of a subgroup H by an element $g \in G$ is denoted by H^g, that is $H^g = gHg^{-1}$.

Definition 2.4. Let H be a subgroup of a group G. Then the *commensurator* of H in G is the set

$$\{g \in G;\ |H : (H \cap H^g)| < \infty,\ |H^g : (H \cap H^g)| < \infty\}.$$

We denote the commensurator by $\mathrm{comm}_G(H)$.

Kapovich and Short [16] call the commensurator the *virtual normalizer*. The following lemma is easy but not entirely obvious, so we include the proof.

Lemma 2.5. *For any group G and subgroup H, the commensurator $\mathrm{comm}_G(H)$ is a subgroup. Further, if $|G : H| < \infty$, then $\mathrm{comm}_G(H) = G$.*

Proof. The second assertion is clear. We prove that $\mathrm{comm}_G(A)$ is a subgroup. We thank Dave Johnson for simplifying our arguments.

We note that if A, B, H are subgroups of G with $A \subset B$, then $|B : A| < \infty$ implies $|(B \cap H) : (A \cap H)| < \infty$. To see this let $x, y \in B \cap H$. Then $x^{-1}y \notin A \cap H$ if and only if $x^{-1}y \notin A$. This says that two cosets $xB \cap H$ and $yB \cap H$ are distinct if and only if the cosets xA and yB are distinct. Then $|(B \cap H) : (A \cap H)| \le |B : A|$.

Let H be a subgroup of G and $x, y \in \mathrm{comm}_G(H)$. Then $|H : (H \cap H^x)| < \infty$ and $|H : (H \cap H^y)| < \infty$. Conjugating the first of these by y, we obtain

$$|H^y : (H^y \cap H^{xy})| < \infty.$$

Now apply the observation above with $H = H$, $B = H^y$ and $A = H^y \cap H^{xy}$. Then $|(H \cap H^y) : (H \cap H^y \cap H^{xy})| < \infty$. From $|H : (H \cap H^y)| < \infty$, it follows that $|H : (H \cap H^y \cap H^{xy})| < \infty$, and then we obtain $|H : H^{xy}| < \infty$. Therefore, $xy \in \mathrm{comm}_H(G)$. It is clear that $1 \in \mathrm{comm}_G(H)$, and if $x \in \mathrm{comm}_G(H)$, then so is x^{-1}. This completes the proof of the lemma. □

Proof of Theorem 2.3. Suppose that A and B are finitely generated commensurable subgroups of a free group F. Then $A \cap B$ has finite index in both A and B and is also finitely generated. We want to show that $A \cap B$ has finite index in the join $\langle A, B\rangle$. Since the join is also a free group, we may, without loss of generality, assume that the join $\langle A, B\rangle$ is all of F.

Since $|A : (A \cap B)| < \infty$, we have $A \subseteq \mathrm{comm}_F(A \cap B)$. To see this, note that by Lemma 2.5, $\mathrm{comm}_A(A \cap B) = A$, but clearly $\mathrm{comm}_A(A \cap B) \subseteq \mathrm{comm}_F(A \cap B)$. Similarly, $B \subseteq \mathrm{comm}_F(A \cap B)$. Since $\mathrm{comm}_F(A \cap B)$ is a subgroup, we have $\langle A, B\rangle \subseteq \mathrm{comm}_F(A \cap B)$, and therefore $F = \langle A, B\rangle = \mathrm{comm}_F(A \cap B)$ again by Lemma 2.5.

Since $A \cap B$ is finitely generated by Hall's theorem, $A \cap B$ is virtually a free factor of F. Hence, there exists a subgroup H of finite index in F such that

$$H = (A \cap B) \star K.$$

However, for any non-trivial $k \in K$, we have $(A \cap B) \cap (A \cap B)^k = \{1\}$. This follows from elementary properties of free products (for all such properties, refer to [28] or [27]). Since $A \cap B$ is infinite, it follows that no non-trivial element of K can be in the commensurator. Since the commensurator is all of F, we must then have $K = \{1\}$ and therefore $H = A \cap B$ has finite index in the join. □

We now present some other results on finitely generated subgroups of free groups that can be proved in the same manner.

Proposition 2.6. *Let H be a finitely generated subgroup of a free group F. Then H has finite index if and only if for each $f \in F$, there exists n such that $f^n \in H$.*

Proof. If $|F : H| = m < \infty$ and $f \in F$, then $1, f, f^2, \ldots, f^m$ cannot all be incongruent modulo H. Therefore, $f^k \in H$ for some k.

Conversely, suppose that for each $f \in F$, we have $f^n \in H$ for some n. Since H is finitely generated, there exists a subgroup K of finite index in F such that $K = H \star K_1$. However, again by elementary properties of free products, no non-trivial element of K_1 can have a power in H. Therefore, K_1 must be trivial and hence $H = K$ is of finite index. □

Corollary 2.7. *Let H be a finitely generated subgroup of a free group F. Then H has finite index if and only if $H \supset F(X^d)$ for some d, where $F(X^d)$ is the verbal subgroup of F generated by all d-th powers.*

As a consequence of Proposition 2.6 and Property S, we obtain the following, which seems to be difficult to prove directly.

Proposition 2.8. *Let A, B and H be finitely generated subgroups of a free group F with $H \subset A \cap B$. If each element of A and B to a sufficiently high power is in H, then each element of the join $\langle A, B\rangle$ to a sufficiently high power is in H.*

Proof. Since each element of A and B to a sufficiently high power is in H, it follows from Proposition 2.6 that H has finite index in both A and B. Then from Property S, H has finite index in the join $\langle A, B\rangle$. It follows clearly that a sufficiently high power of each element in the join is in H. □

Proposition 2.9. *A finitely generated non-trivial normal subgroup N of a free group F must be of finite index.*

Proof. Suppose that N is normal in F with $|F : N| < \infty$. Since N is finitely generated, there exists a subgroup K of finite index in F such that $K = N \star K_1$. But then as before, $N \cap N^k = \{1\}$ for any non-trivial $k \in K_1$, contradicting the normality. Therefore, K_1 must be trivial and $N = K$ has finite index. □

We note that a version of this result was recently proved by Bridson and Howie [5] for non-abelian limit groups (see Section 4).

Corollary 2.10. *Let H be a finitely generated subgroup of a free group F. Then H has finite index if and only if H contains a finitely generated non-trivial normal subgroup.*

Proof. If $|F : H| < \infty$, then the intersection of the conjugates of H is a finitely generated normal subgroup contained in H. Conversely, suppose that N is a finitely generated normal subgroup of F contained in H. Then from Proposition 2.9, N has finite index and therefore so does H. $\square$

Corollary 2.10 provides a transparent proof of the residual finiteness of free groups.

Corollary 2.11. *A finitely generated free group F is residually finite.*

Proof. Let $g \in F$. From Hall's theorem, there exists a subgroup K of finite index in F such that $K = \langle g \rangle \star K_1$. Since these are free groups, there is clearly a homomorphism ϕ of K onto a finite group G with $\phi(g) \neq 1$. Let N be the kernel of ϕ so that $|K : N| < \infty$ and $g \notin N$. Since $|F : K| < \infty$, we have $|F : N| < \infty$. Take $N_1 = \bigcap_{h \in F} N^h$. Then $|F : N_1| < \infty$. Since $g \notin N$, it follows that $g \notin N_1$, completing the proof. $\square$

The same argument can be extended to the stronger property of subgroup separability. Recall that a group G is *subgroup separable* if given any finitely generated subgroup H of G and $g \notin H$, there exists a finite quotient $\overline{G}$ of G with $\overline{g} \notin \overline{H}$, where $\overline{g}$ and $\overline{H}$ are the respective images of g and H in $\overline{G}$. The Marshall Hall property was used originally to prove subgroup separability of free groups. A proof of this using Hall's theorem is an extension of the proof of the residual finiteness. Clearly, the subgroup separability implies residual finiteness. In the next theorem, we present another proof of Hall's result that free groups are subgroup separable.

Theorem 2.12. *Let F be a free group, H a finitely generated subgroup, and $\{g_1, \ldots, g_n\}$ a finite set of elements of $F \backslash H$. Then there exists a subgroup G of finite index in F, containing H and not containing $\{g_1, \ldots, g_n\}$.*

Proof. Without loss of generality we may assume that F is finitely generated. It is sufficient to prove the theorem for $n = 1$. For arbitrary n, the result will then follow by induction. Indeed, we may assume that there exists a subgroup M of finite index in F that contains $\langle H, g_1, \ldots, g_{n-1} \rangle$ and does not contain g_n. By induction we can find the desired subgroup G in M.

Assume then that $n = 1$. Let $g = g_1$ and $H_1 = \langle H, g \rangle$. From Hall's theorem, there exists a subgroup L in F such that $H_1 * L$ has finite index in F. Hence, we may assume $F = H_1 * L$. Consider the retraction $\phi : F \to H_1$ identical on H_1 and trivial on L. It is sufficient to find a finite index subgroup G_1 in H_1 containing H and not containing g. Then we can set $G = \phi^{-1}(G_1)$.

By Hall's theorem, we have that $H * M$ has finite index in H_1 for some subgroup M. If $H * M$ is a proper subgroup of H_1, we can set $G_1 = H * M$. Now suppose $H * M = H_1$. Consider the retraction $\psi : G_1 \to M$ identical on M and trivial on H. Since $M \neq \{1\}$, we can find a proper normal subgroup M_1 of M of finite index in M which does not contain g. Then we set $G_1 = \psi^{-1}(M_1)$. $\square$

Recently, Wilton [42] has proved that all finitely generated fully residually free groups (i.e., limit groups) are subgroup separable, answering a question of Sela.

The next proposition was proved in [15] using Stallings foldings. The proof we give seems to be more direct.

Proposition 2.13. *Let H be a finitely generated non-trivial subgroup of a free group F. Then H has finite index in its commensurator.*

Proof. Let $F_1 = \mathrm{comm}_F(H)$. Then F_1 is free and H is a finitely generated subgroup of F_1. As before, there exists a subgroup K of finite index in F_1 such that $K = H \star K_1$. No non-trivial element of K_1 can be in the commensurator and hence $K_1 = \{1\}$. Therefore, $K = H$ has finite index in F_1. □

Corollary 2.14. *A finitely generated non-trivial subgroup H of a free group F has finite index if and only if $F = \mathrm{comm}_F(H)$.*

In general, if G is any group and $|G : H| < \infty$, then $G = \mathrm{comm}_G(H)$. However, the converse need not be true. As a counterexample, consider the amalgamated free product G of two finite groups G_1, G_2 with amalgamated finite subgroup H with $G_1 \neq H \neq G_2$. Here $G = G_1 \star_H G_2$. Then $\mathrm{comm}_G(H) = G$ but G is infinite, so H has infinite index.

Kapovich and Myasnikov [15] proved Corollary 2.14 using a graph-theoretic argument based on Stallings foldings. They then use Corollary 2.14 to prove Property S for free groups. Abstracting some of these ideas, we define:

Definition 2.15. A group G has the *commensurator condition* if any finitely generated non-trivial subgroup has finite index in its commensurator.

Let G be a group satisfying the commensurator condition. Then a finitely generated non-trivial subgroup H of G has finite index if and only if its commensurator within G is all of G. The following is then easy.

Proposition 2.16. *If a group G satisfies the commensurator condition, then it satisfies Property S.*

The converse is not true. An infinite finitely generated abelian group provides a counterexample (see Section 6). In Theorem 4.7, we use the commensurator condition to give a separate proof of a result of Kapovich that hyperbolic limit groups satisfy Property S.

We close this section with the following technical proposition also based on Hall's theorem. This is useful in the study of one-relator groups (see [9]).

Proposition 2.17. *Let $F = \langle a_1, \ldots, a_p \rangle$ be a free group of rank p with basis $\{a_1, \ldots, a_p\}$. Let w_1 be an element of F which is neither a proper power nor primitive in F. Let $w = w_1^k$ with k a natural number and N the normal closure of w in F. Let H be a free subgroup of F with rank p and basis $\{x_1, \ldots, x_p\}$. Suppose $w_1^\alpha \in \langle x_1, \ldots, x_n \rangle$ with $1 \leq n < p$ for some $\alpha \geq 1$ and further $F = HN$. Then $w_1 \in \langle x_1, \ldots, x_n \rangle$ and $F/N \cong \langle x_1, \ldots, x_n;\, w = 1 \rangle \star K$, where $K = \langle x_{n+1}, \ldots, x_p \rangle$.*

Proof. By assumption F/N has rank p. Let $\alpha \geq 1$ be the minimal power such that $w_1^\alpha \in \langle x_1, \ldots, x_n \rangle$ and assume $\alpha \geq 2$. Since $F/N \cong HN/N \cong H/(N \cap H)$ has rank p, there is no conjugate of a power of w_1 primitive in $\langle x_1, \ldots, x_n \rangle$. Let $R(x_1, \ldots, x_n) = w_1^\alpha$ with $R(x_1, \ldots, x_n)$ not a proper power in $\langle x_1, \ldots, x_n \rangle$. As a subgroup of F, the group G generated by $w_1, x_1, \ldots, x_n$ is also free. Then by [1], G

has rank $n-1$ which contradicts $F = HN$. Therefore, $w_1 \in \langle x_1, \ldots, x_n \rangle$. Suppose first that H is of finite index in F. Let $v \in N$ with $v \notin H$. Then $v^\beta \in H$ for some $\beta \geq 2$. No conjugate of a power of v is primitive in H because $F/N \cong H/(N \cap H)$. Consider the subgroup M of F generated by $v, x_1, \ldots, x_p$. Again from [1], we have that M has rank $\leq p-1$. This contradicts that

$$F/N \cong H/(N \cap H) \cong M/(N \cap M)$$

has rank p. Therefore, $N \subset H$ if H has finite index and the result holds.

Now suppose that H has infinite index in F. Then from Hall's theorem, there is a subgroup H_1 in F such that $H \star H_1$ has finite index in F. Let $v \in N$ with $v \notin H \star H_1$. As before, $v^\beta \in H \star H_1$ for some $\beta \geq 2$. Now consider the subgroup Q generated by v and $H \star H_1$. As in the previous case, we get a contradiction unless v^β is contained in the normal closure of H_1 in $H \star H_1$. However, in this case, we would have $Q = H \star H_2$ and the index of $H \star H_2$ in F is smaller than the index of $H \star H_1$ in F. Therefore, by induction we may assume that there is a subgroup P of F such that $H \star P$ is a subgroup of F with finite index and $N \subset H \star P$. This gives the desired result. □

3 Marshall Hall Groups and Property *S*

Brunner and Burns [7] considered groups which satisfy the conclusion of Hall's theorem. In particular:

Definition 3.1. A group G is a *Marshall Hall group* if any finitely generated subgroup H of G is virtually a free factor of G.

In particular, free groups are Hall groups. Using the Stallings theory of ends of groups, Brunner and Burns proved the following two theorems.

Theorem 3.2. [7] *A freely indecomposable finitely generated Hall group is cyclic, finite, a proper amalgamated product over a finite group, or an HNN extension over a finite group.*

Theorem 3.3. [7] *Suppose that A and B are finite groups with $A \cap B = U$ malnormal in at least one of A and B. Then the amalgamated product $G = A \star_U B$ is a Hall group.*

Corollary 3.4. *A finitely generated Hall group is virtually free. Therefore, a torsion-free Hall group is free.*

The class of finitely generated virtually free groups coincides with the class of groups isomorphic to the fundamental group of a finite graph of finite groups (see [17] and [39, pp. 120–121]). The corollary then follows from this fact and Theorem 3.2 together with the fact that Hall's property is inherited by subgroups (see [7]). Therefore, we also have the following.

Corollary 3.5. *A finitely generated Hall group is isomorphic to the fundamental group of a finite graph of finite groups.*

Not every virtually free group is a Hall group. For example, the group $\mathbb{Z}_4 \star_{\mathbb{Z}_2} \mathbb{Z}_4$ or more generally any group which does not satisfy Property $\mathcal{N}$ is not a Hall group. A group G satisfies Property $\mathcal{N}$ if for every finitely generated subgroup H, the index of H in its normalizer is finite (see [4]). Thus, it is natural to ask the question as to which virtually free groups are actually Hall groups. Bogopolski answered this question.

Theorem 3.6. [4] *Let G be a fundamental group of a finite graph of finite groups. Then G is a Hall group if and only if any vertex group is a free factor in a subgroup of finite index in G. Further, given a finite graph of finite groups, we can algorithmically decide whether its fundamental group is a Hall group or not.*

In [7] it was claimed that Theorem 3.3 was "if and only if". However, Bogopolski in [4] showed that the condition that C is malnormal in at least one of A and B is sufficient but not necessary. The following (see [4]) provides a counterexample. Set $A = \langle x_1, y_1\rangle \times \langle z_1\rangle = S_3 \times \mathbb{Z}_2$ and $B = \langle x_2, y_2\rangle \times \langle z_2\rangle = S_3 \times \mathbb{Z}_2$, where x_1, x_2 are elements of order 3 in S_3 and y_1, y_2, z_1, z_2 are elements of order 2. Let C_1 be the subgroup of A generated by z_1, y_1. Let C_2 be the subgroup of B generated by z_2, y_2. Clearly, $C_1 \cong C_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Define a specific isomorphism $\phi : C_1 \to C_2$ by the rule $\phi(z_1) = z_2$ and $\phi(y_1) = y_2$. Now consider the amalgamated product

$$G = \langle A, B;\, c = \phi(c),\, c \in C_1\rangle \cong A \star_C B$$

with $C \equiv C_1$. Then it was shown in [4] that G is a Hall group but C is malnormal in neither A nor B.

In general, Hall groups do not satisfy the strong form of Property S as defined in the last section. Consider a Hall group G given by $G = A \star_U B$, where A, B are finite and U is malnormal in one of them with $A \neq U \neq B$. Since $A \cap B = U$, A and B are commensurable in G, however, U does not have finite index in the join which is all of G. Although G satisfies Hall's property, the proof of Theorem 2.3 breaks down in that $A \cap B$ is finite so that $|(A \cap B) : (A \cap B) \cap (A \cap B)^g| < \infty$ for all $g \in G$. However, this cannot happen if we restrict A, B to be infinite.

Definition 3.7. A group G has *Property S_1* if given finitely generated commensurable infinite subgroups A and B of G, then $A \cap B$ has finite index in $\langle A, B\rangle$.

Theorem 3.8. *Hall groups satisfy Property S_1.*

Every Hall group is virtually free. However, not every virtually free group is a Hall group, though Property S_1 is satisfied by every virtually free group.

Theorem 3.9. *Any virtually free group G satisfies Property S_1. In particular, any virtually Hall group satisfies Property S_1.*

Proof. Suppose that $|G : F| < \infty$ with F free and A, B are finitely generated infinite commensurable subgroups of G. Since $|A : (A \cap B)| < \infty$ and $|B : (A \cap B)| < \infty$, it follows that $A \subset \mathrm{comm}_G(A \cap B)$ and $B \subset \mathrm{comm}_G(A \cap B)$. Therefore, $\langle A, B\rangle \subset \mathrm{comm}_G(A \cap B)$, and hence $\langle A, B\rangle \cap F \subset \mathrm{comm}_G(A \cap B)$. Now let $H_1 = \langle A, B\rangle \cap F$.

Then clearly $\mathrm{comm}_{H_1}(A \cap B) = \langle A, B\rangle \cap F$. However, then

$$\mathrm{comm}_{H_1}(A \cap B \cap F) = (\mathrm{comm}_G(A \cap B)) \cap H_1 = \langle A, B\rangle \cap F.$$

Since $A \cap B \cap F$ has finite index in $A \cap B$, it follows that $A \cap B \cap F$ is finitely generated. Moreover, since F is a free group, it follows that $A \cap B \cap F$ and $\langle A, B\rangle \cap F$ are also free groups, so from Corollary 2.14, $A \cap B \cap F$ has finite index in $H_1 = \langle A, B\rangle \cap F$. Since F has finite index in G, it follows that H_1 has finite index in $\langle A, B\rangle$. Putting these together we get that $A \cap B$ has finite index in $\langle A, B\rangle$. □

4 Surface Groups and Property *S*

Kapovich and Short have proved the following restricted version of Property S for torsion-free hyperbolic groups.

Theorem 4.1. [16] *Suppose that H is a torsion-free hyperbolic group, and A, B are finitely generated, commensurable, quasi-convex subgroups of H. Then $A \cap B$ has finite index in $\langle A, B\rangle$.*

We call a hyperbolic group *QC-free* if any finitely generated subgroup is quasi-convex. Free groups and hyperbolic surface groups are QC-free. Recall that a surface group G is hyperbolic if $G = \langle a_1, b_1, \ldots, a_g, b_g;\ [a_1, b_1] \cdots [a_g, b_g] = 1\rangle$ with $g \geq 2$ or $G = \langle a_1, \ldots, a_g;\ a_1^2 \cdots a_g^2 = 1\rangle$ with $g \geq 3$. For general finitely generated groups, the QC-free property has been called locally quasi-convex.

Corollary 4.2. *Any QC-free hyperbolic group satisfies Property S.*

Corollary 4.3. *Any hyperbolic surface group satisfies Property S.*

This result can be rephrased in the following interesting way.

Theorem 4.4. *Let G be a hyperbolic surface group. Suppose that A and B are finitely generated commensurable free subgroups of G. Then the join $\langle A, B\rangle$ is a free group.*

Proof. Suppose that A and B are finitely generated commensurable free subgroups of the hyperbolic surface group G. It is known (see for example [10]) that subgroups of finite index in G are also surface groups of finite genus while subgroups of infinite index are free. Hence, the join $\langle A, B\rangle$ is either a surface group of finite genus or a free group. From the Greenberg–Kapovich–Short result, the intersection $A \cap B$ has finite index in $\langle A, B\rangle$. $A \cap B$ is a free group. If $\langle A, B\rangle$ is a surface group of finite genus, then it would have a free group of finite index which is impossible. Hence, $\langle A, B\rangle$ must be free. □

We can further extend this.

Corollary 4.5. *Suppose that A and B are finitely generated commensurable free subgroups of $PSL_2(\mathbb{R})$. If the join $\langle A, B\rangle$ is discrete, it must also be free.*

The proof of Kapovich and Short uses the boundaries of hyperbolic groups and, of course, quasi-convexity. This raises the following question:

Question 4.6. Is there a purely algebraic or purely function-theoretic proof of the results given in Theorem 4.4 and Corollary 4.5?

Given our proof of Property S for free groups using Hall's theorem, this raises the question of whether there is an analog of Hall's theorem for surface groups. The answer is yes for hyperbolic surface groups of even genus. We obtain the following result. However, it does not allow for a proof of Property S to go through.

Theorem 4.7. *Let G be a hyperbolic surface group of even genus. Let $A = \{a_1, \dots, a_n\}$ be a finite set of elements of G. Then there exists a subgroup H of finite index in G such that A is included in a generating system for H.*

Proof. We prove the theorem in the orientable case. The proof for the non-orientable case is analogous. Let K be an orientable surface group of genus $2k$ so that

$$K = \langle x_1, \dots, x_{2k}, y_1, \dots, y_{2k};\ [x_1, x_2] \cdots [x_{2k-1}, x_{2k}] = [y_1, y_2] \cdots [y_{2k-1}, y_{2k}] \rangle.$$

Now consider A to be free on $4k$ generators $x_1, \dots, x_{2k}, y_1, \dots, y_{2k}$ and let ϕ be the involution which transposes the x_i and y_i. Let $G = \langle t, A;\ t^{-1}at = \phi(a),\ a \in A \rangle$.

Recall that a finitely generated subgroup H of a group G has the *generalized Hall's property* if whenever $g_0 \notin H$ there is a subgroup $H^\star$ of finite index in G such that $g_0 \notin H^\star$ and there is a graph decomposition of groups for $H^\star = \pi_1(\Gamma, \mathcal{H})$ with $\mathcal{H}(v) = H \star K$ for some vertex group $\mathcal{H}(v)$.

From a theorem of Tretkoff [41], the HNN group G given above satisfies the generalized Hall's property, where $a = [x_1, x_2] \cdots [x_{2k-1}, x_{2k}]$. Further, if $z_i = t^{-1}x_i t$, then K embeds into G by $x_i \mapsto y_i$ and $y_i \mapsto z_i$.

Let H be a finitely generated subgroup of K considered as being a subgroup of G. Clearly, $t \notin H$. Then from the generalized Hall's property, there is a subgroup G_1 of G such that

(1) $G_1 = \pi_1(\Delta, \mathcal{H})$ for some graph of groups $(\Delta, \mathcal{H})$,

(2) for some vertex v of Δ, we have $\mathcal{H}(v) = H \star L$,

(3) $t \notin G_1$.

Now $H \leq G_1 \cap K \leq K$ and $G_1 \cap K$ has finite index in K. Further, since $\mathcal{H}(v)$ is a free product with H as a factor, a generating system for H is part of a generating system of $\mathcal{H}(v)$. In turn, $\mathcal{H}(v)$ is a vertex group of a graph of groups, so its generating system is part of a generating system for G_1. Therefore, a given generating system for H is part of a generating system for a subgroup of finite index in K. □

Let G be a hyperbolic surface group. A generating system X for G is called a *standard generating system* if X is a generating system over which G splits as a free product with cyclic amalgamation. The problem in applying Theorem 4.7 to proving Property S for hyperbolic surface groups of even genus is that the generating system that we obtain for A to be part of is not guaranteed to be a standard generating system with A in one of the factors. We pose the question of when we can extend

Theorem 4.7 so that A is part of a standard generating system and contained in one of the factors expressing G as a cyclically pinched one-relator group. One straightforward condition is when A is actually part of X.

Proposition 4.8. *Let X be a standard generating system for a hyperbolic surface group G. Then given $Y \subset X$, there exists a proper subgroup of finite index H with a decomposition $H = G_1 \star_C G_2$ such that Y is part of a generating system for G_1.*

Proof. We prove this proposition only in the orientable case and where G has genus 2. The higher genus case and the non-orientable case are done in an analogous manner. Then let G have the presentation $G = \langle a, b, c, d;\ [a, b] = [c, d]\rangle$. The subsets $\{a, b\}$ and $\{c, d\}$ are already part of standard generating systems for the whole group G. Consider then the subset $\{a, b, c\}$ of the standard generators and we show that it is part of a standard generating system for a subgroup of finite index. Let K be the normal closure of $\langle a, b, c, d^2\rangle$ in G. The quotient is $G/K = \langle d;\ d^2 = 1\rangle$. Hence, K has index 2 in G. Taking the elements $1, d$ as coset representatives for K and applying the Reidemeister–Schreier rewriting process, we obtain a presentation $K = \langle a, b, c, \delta, \alpha, \beta;\ [a, b][c, \delta][\beta, \alpha] = 1\rangle$, where $\delta = d^2$, $\alpha = da^{-1}d^{-1}$ and $\beta = db^{-1}d^{-1}$. If we let $G_1 = \langle a, b, c, \delta\rangle$ and $G_2 = \langle \alpha, \beta\rangle$, then $K = G_1 \star_C G_2$ with $C = \langle [a, b][c, \delta]\rangle_{G_1} = \langle [\beta, \alpha]^{-1}\rangle_{G_2}$ and the set $\{a, b, c\}$ is part of the generating system for G_1. □

In another direction, Kapovich [14] has proved that all fully residually free groups satisfy Property S. This also shows that hyperbolic surface groups satisfy Property S (of genus ≥ 4 in the non-orientable case). Recall that a group G is *fully residually free* if given finitely many non-trivial elements $g_1, \ldots, g_n$ in G, there is a homomorphism $\phi : G \to F$, where F is a free group such that $\phi(g_i) \neq 1$ for all $i = 1, \ldots, n$. Fully residually free groups have played a crucial role in the study of equations and first order formulas over free groups and in the solution of the Tarski problem by Kharlampovich and Myasnikov [18–23] and independently Sela [32–38]. A theorem due to Remeslennikov [31] and independently Gaglione and Spellman [11] shows that the finitely generated non-abelian fully residually free groups coincide with the finitely generated *universally free groups*, that is the class of groups having the same universal (equivalently existential) theory as the class of non-abelian free groups. The structure and properties of fully residually free groups have been extensively studied by Kharlampovich and Myasnikov (see [22] and the references there). The term *limit group* was introduced by Sela (see [6]) for finitely generated fully residually free groups to emphasize the fact that these are precisely the class of groups that arise when one takes limits of stable sequences of homomorphisms $\phi_n : G \to F$, where G is an arbitrary finitely generated group and F is free. It is known that a limit group is hyperbolic precisely when all centralizers are cyclic. Wilton [42] has recently proved that all limit groups are subgroup separable. Using the concept of Stallings foldings for infinite words, we can give a separate proof of Kapovich's result for hyperbolic limit groups — it is a nice illustration of the technique.

Theorem 4.9. *A hyperbolic limit group satisfies the commensurator condition, that is, any finitely generated non-trivial subgroup is of finite index in its commensurator.*

Proof. Recall that a hyperbolic limit group can be viewed as a finitely generated subgroup of $F^{\mathbb{Z}[t]}$, the free $\mathbb{Z}[t]$-group (see [29, 30]). The hyperbolicity ensures that all non-trivial abelian subgroups are infinite cyclic. In [29, 30] it was shown that the elements of $F^{\mathbb{Z}[t]}$ can be viewed as *infinite words* defined in the following manner. Let A be a discretely ordered abelian group, i.e., an ordered abelian group with a least positive element denoted by $1_A \in A$. Let X be a set. Closed intervals are defined in A in the usual way. Then an *A-word* or *infinite word* is a function

$$w : [1_A, \alpha_w] \to X^{\pm 1},$$

where $\alpha_w \in A$ with $\alpha_w \geq 0$. The element α_w is called the length of w denoted as usual by $|w|$. The infinite word w is reduced if $w(\alpha) \neq w(\alpha + 1)^{-1}$ for any $1 \leq \alpha \leq \alpha_w$. We denote the set of all infinite words by $W(A, X)$. The basic combinatorial techniques for words in free groups, reduction, inversions, etc. can be carried over to infinite words. In particular, subgroups of $F^{\mathbb{Z}[t]}$ have representations as sets of reduced infinite words in $W(A, X)$ with $A = \mathbb{Z}[t]$ the integral polynomial ring, the order being the standard lexicographic order and X a set of generators for the subgroup. For details, see [29, 30].

Using infinite words, many algorithmic and combinatorial problems in fully residually free groups can be solved using analogous, often identical, methods as those methods used to solve the same problems in free groups (see [24]). In particular, there is an analog of the Stallings subgroup graph and the folding method. If the abelian subgroups are all cyclic, as in the case of hyperbolic limit groups, the proofs carry through almost identically.

The proof of Theorem 4.9 is entirely analogous to the proof, of what we term here the commensurator condition, given in [15] for the case of free groups where infinite words are used instead of ordinary words. We refer to [24] for terminology.

A Stallings subgroup graph Γ labeled by a generating set X is called X-regular if for every vertex $v \in \Gamma$ and every $x \in X \cup X^{-1}$, there is exactly one edge in the oriented graph of Γ (see [15]) with origin v and label x. It was proved in [15] and carried over to fully residually free groups that a subgroup H is of finite index if and only if its subgroup graph $\Gamma(H)$ is a finite X-regular graph.

Now let G be a hyperbolic limit group and H a non-trivial finitely generated subgroup. We must show that $\mathrm{comm}_G(H) = G$ if and only if H is of finite index. From the construction of fully residually free groups of low rank (see [8]), the result follows easily if rank G is 2 or less. Therefore, we may assume that G has rank at least 3. If $|G : H| < \infty$, then it is clear that $\mathrm{comm}_H(G) = G$, so we must prove the converse. Suppose $\mathrm{comm}_G(H) = G$.

Assume $|G : H| = \infty$. Suppose that X is a finite generating system for G (recall that hyperbolic groups are finitely presented). Then the subgroup graph $\Gamma(H)$ labeled by X is not X-regular. Hence, there exists a vertex v and a letter $x \in X \cup X^{-1}$ such that there is no edge labeled x whose origin v is in $\Gamma(H)$. Since $\operatorname{rank} G$ is greater than 2, we can choose $a\overline{x}$ with $a \neq x^{\pm 1}$. Since the commensurator of H is all of G, for any $g \in G$ and any $h \in H$, we have a power $n \geq 1$ such that $g^{-1}h^n g \in H$. This follows since for any $g \in G$ we have $|gHg^{-1} : (H \cap gHg^{-1})| < \infty$, and hence for any element of gHg^{-1}, there is a power in $H \cap gHg^{-1}$ and hence in H.

Let $h \in H$ be given as a reduced infinite word in X. Let $y \in X \cup X^{-1}$ be the first letter of h and $z \in X \cup X^{-1}$ be the last letter of h. By the free reduction of infinite words, the reduced form of h^n for any $n \geq 1$ also begins with y and ends with z. Let $\overline{h^n}$ be the reduced form of h^n. Let w be the label of a reduced path in $\Gamma(H)$ from 1_H to v. Since there is no path labeled x with origin v in $\Gamma(H)$, any reduced infinite word with initial segment wx is not in H.

Let $q = y$ if $y \neq z^{-1}$. If $y = z^{-1}$ and $y \in \{x, x^{-1}\}$, let $q = a$. If $y = z^{-1}$ and $y \notin \{x, x^{-1}\}$, let $q = x$. From these choices, it follows that for any power $m \geq 1$, the infinite word $q\overline{h^m}q^{-1}$ is reduced.

Now choose an infinite reduced word w' such that $wxw'q$ is also reduced. This choice is possible since rank $G > 2$. Let $g = wxw'q$. Then as explained above, there is a power $n \geq 1$ such that $gh^ng^{-1} \in H$. However, from the choice of q, the reduced form of gh^ng^{-1} is $wxw'q \cdots q^{-1}(w')^{-1}x^{-1}w^{-1}$. This contradicts that a path in $\Gamma(H)$ begins with wx. Therefore, the graph cannot be not X-regular and hence $|G : H| < \infty$.

From the arguments in [29, 30], if centralizers were not cyclic, some of the choices would not be possible. □

Corollary 4.10. *Any hyperbolic limit group satisfies Property S.*

As mentioned, Kapovich [14] has proved the whole result, that is, all fully residually free groups and hence limit groups satisfy Property S.

5 Nilpotent Groups, Property *R* and Property *S*

As a consequence of a result of Mal'cev (see [2]), Property S can be shown to hold in all finitely generated nilpotent groups. Mal'cev proved the following. A proof can be found in [2].

Theorem 5.1. (Mal'cev) *Let G be a finitely generated nilpotent group and H a subgroup of G. Then H is of finite index in G if and only if some positive power of each element of a set of generators for G lies in H. Further, in this case, a positive power of every element of G lies in H.*

We use the conclusion of Mal'cev's result to define what we term Property R.

Definition 5.2. Let G be a finitely generated group generated by the finite set X. G satisfies Property $\tilde{R}_X$ (with respect to X) if any subgroup H is of finite index if and only if some positive power of each element of X lies in H. G satisfies Property $\tilde{R}$ if it satisfies Property $\tilde{R}_X$ for each finite generating system X of G.

Let G have Property $\tilde{R}$ and let H be a finitely generated subgroup. Then H has finite index if and only if for each $g \in G$, there exists a positive power n such that $g^n \in H$. In particular, any finitely generated nilpotent group has Property $\tilde{R}$.

Definition 5.3. Let G be a finitely generated group with Property $\tilde{R}$. Then G has Property R if in addition each finitely generated subgroup of G has Property $\tilde{R}$.

Property R is trivially true in finite groups, so it constitutes another finiteness property for infinite groups. In this setting, Mal'cev's result can be expressed as:

Theorem 5.4. (Mal'cev) *Finitely generated nilpotent groups satisfy Property R.*

A proof of Theorem 5.4 is given in [2] and in Kurosh's book [25].

Theorem 5.5. *If a finitely generated group G satisfies Property R, then G also satisfies Property S.*

Proof. Suppose that G is a finitely generated group satisfying Property R. Let A and B be finitely generated commensurable subgroups of G. Let X be a finite generating system for A and Y a finite generating system for B. Then $A \cap B$ has finite index in both A and B. Since A and B are finitely generated, it follows from Property R that there is a large enough power n such that $x^n \in A \cap B$ for each $x \in X$. Similarly, there is a large enough power m such that $y^m \in A \cap B$ for each $y \in Y$. Hence, there is a large enough power k such that $x^k \in A \cap B$ and $y^k \in A \cap B$ for each $x \in X$ and $y \in Y$. However, the join $\langle A, B\rangle$ is generated by the union of the generators of A and B. Since the join is finitely generated, it satisfies Property R and therefore $A \cap B$ has finite index in the join $\langle A, B\rangle$. □

We note that the converse of Theorem 5.5 is not true. As an example, consider the free group $F = \langle a, b\rangle$ of rank 2, which satisfies Property S. However, let H be the normal subgroup of F generated by a^2, b^2. Then $F/H = \langle a, b;\ a^2 = b^2 = 1\rangle$ is the infinite dihedral group. It follows that H has infinite index in F even though $a^2 \in H$ and $b^2 \in H$.

Now directly from Mal'cev's result, we have:

Corollary 5.6. *Finitely generated nilpotent groups satisfy Property S.*

The following are easy consequences of Property R and mirror results in free groups (see Section 2). Of course, the proofs cannot depend on Hall's theorem as they did in the free group case.

Proposition 5.7. *Let G have Property R and let H be a finitely generated subgroup. Then H has finite index if and only if for each $g \in G$, there exists n such that $g^n \in H$. In particular, any finitely generated nilpotent group has this property.*

Corollary 5.8. *Let H be a finitely generated subgroup of a group G having Property R. Then H has finite index if and only if $H \supset G(X^d)$ for some d, where $G(X^d)$ is the verbal subgroup of G generated by all d-th powers.*

Proposition 5.9. *Let A, B and H be finitely generated subgroups of a group G with Property R with $H \subset A \cap B$. If each element of A and B to a sufficiently high power is in H, then each element of the join $\langle A, B\rangle$ to a sufficiently high power is in H. In particular, this holds in finitely generated nilpotent groups.*

Proof. The proof is the same as in free groups (see Proposition 2.8). Since each element of A and B to a sufficiently high power is in H, it follows that H has finite index in both A and B. Then from Property S, H has finite index in the join $\langle A, B \rangle$. It follows clearly then that a sufficiently high power of each element in the join is in H. □

We note that neither Property R nor Property S is preserved under general free products or general finite extensions. The infinite dihedral group provides a counterexample in each case. Suppose $G = \mathbb{Z}_2 \star \mathbb{Z}_2$. Then since both factors are finite, they satisfy both Property R and Property S. Further, they are commensurable since they are finite, the intersection being the identity. However, the join is the whole group which is infinite, so the intersection does not have finite index in the join. Therefore, $\mathbb{Z}_2 \star \mathbb{Z}_2$ does not satisfy Property S. Similarly, each generator to the power 2 is the identity but the identity does not have finite index, so it also does not satisfy Property R.

Further, $\mathbb{Z}_2 \star \mathbb{Z}_2$ is virtually infinite cyclic. Infinite cyclic groups satisfy both Property R and Property S, and hence these properties are not preserved under arbitrary finite extensions.

This raises the question of how to classify the finite extensions of Property R groups which still satisfy Property R. Certain finite extensions of Property R groups do still satisfy Property R.

Theorem 5.10. *Let G satisfy Property R and let H be an arbitrary finite group. Then $H \times G$ satisfies Property R.*

Proof. Let $X = \{(h_1, g_1), \ldots, (h_m, g_m)\}$ be a finite generating system of $H \times G$ and K a subgroup of $H \times G$. Suppose that for any (h_i, g_i), a sufficiently large power lies in K. From the assumed property for K, there is an integer n such that

$$(h_i, g_j)^n = (h_i^n, g_j^n) \in K$$

for all i, j. Since H is a finite group, there is a power m such that $h_i^m = 1$ for all i. Therefore, there is a positive integer t such that $(1, g_j^t) \in K$ for each generator g_j of G. Since $(1, g_j^t) \in K \cap \overline{G}$, where $\overline{G} = 1 \times G \equiv G$ and $\overline{G}$ has Property R, it follows that $K \cap \overline{G}$ has finite index in G. However, since H is finite, G has finite index in $H \times G$ and therefore $K \cap G$ has finite index in $H \times G$. It follows that K must have finite index in $H \times G$.

If K_1 is a finitely generated subgroup of K, then K_1 is contained in $H_1 \times G_1$, where G_1 is the projection of K into G and H_1 is the projection of K into H. Suppose that K_1 is a subgroup such that each generator of K to a sufficiently high power is in K_1. Then since H_1 is finite, there is a sufficiently high power of each generator of $G_1 \times H_1$ in K_1. Since G_1 is finitely generated, it satisfies Property R, so from the first part of the proof, K_1 has finite index in $G_1 \times H_1$ and hence in K, completing the proof. □

6 A Non-Virtually Nilpotent Property *R* Group

In the previous section, the examples of groups with Property R all were virtually nilpotent. In this section, we present a class of groups each of which has Property R

and hence Property S, but which are not virtually nilpotent. For each integer $n \geq 2$, let

$$B_n = \langle a, t;\ t^{-1}at = a^n \rangle.$$

Our main result is the following:

Theorem 6.1. *For each $n \geq 2$, the group B_n satisfies Property R and hence Property S. Further, each B_n is not virtually nilpotent.*

We show that B_n satisfies Property R and therefore Property S. We prove it first for $n = 2$. The cases $n > 2$ then follow from the same arguments. The proof depends on the following proposition which is of interest in its own right.

Proposition 6.2. *The non-abelian subgroups of $B_2 = \langle a, t;\ t^{-1}at = a^2 \rangle$ are all of finite index.*

This will be proved by a succession of lemmas. The first is a standard result on finite cyclic groups (see [12]).

Lemma 6.3. *Let A be a cyclic group of order n, where $n > 1$. Then $\phi : x \mapsto x^m$ is an automorphism of A whenever n and m are relatively prime.*

Corollary 6.4. *Let A be a cyclic group of order n and let m be a positive integer relatively prime to n. Then $G = \langle a, t;\ a^n = 1,\ t^{-1}at = a^m \rangle$ is an extension of a cyclic group of order n by an infinite cyclic group.*

Now let H be the commutator subgroup of B_2. The commutator subgroup of B_2 is the normal closure of $[a, t]$. However, in B_2 we have $[a, t] = a$. Hence, H is the normal closure of the element a in B_2. Observe that B_2/H is infinite cyclic and is generated by tH. It follows either by the method of Reidemeister–Schreier or by Magnus' method for solving the word problem for one-relator groups that if we put $a_i = t^{-i}at^i$ ($i \in \mathbb{Z}$), then H can be presented as

$$H = \langle \ldots, a_{-1}, a_0, a_1, \ldots ;\ \ldots, a_{-1}^2 = a_0,\ a_0^2 = a_1, \ldots \rangle.$$

So for every $i \in \mathbb{Z}$, we find $a_i^2 = a_{i+1}$. It follows that H is a multiplicative copy of the additive subgroup of $\mathbb{Q}$ consisting of those rational numbers of the form $\ell/2^m$, where ℓ and m range over $\mathbb{Z}$ (see [25, vol. 1]). If we now use the obvious exponential notation to represent the elements of H, this allows us to denote the pre-image of $\ell/2^m$ in H by $a^{\ell/2^m}$. Using this notation, we find $t^{-1}(a^{\ell/2^m})\,t = a^{\ell/2^{m-1}}$ and $t(a^{\ell/2^m})\,t^{-1} = a^{\ell/2^{m+1}}$. If we now invoke this notation, it is easy to prove the following.

Lemma 6.5. *Let J be a subgroup of H and let k be a fixed positive integer. If $t^k x t^{-k} \in J$ for every $x \in J$, then J is a normal subgroup of B_2.*

Proof. Since H is abelian, in order to prove this lemma, it suffices to prove that if $x \in J$, then $txt^{-1} \in J$ since $t^{-1}xt = x^2$. Now $x = a^{\ell/2^m}$ with $\ell, m \in \mathbb{Z}$. By hypothesis, $t^k x t^{-k} = a^{\ell/2^{m+k}} \in J$. Consequently, $(a^{\ell/2^{m+k}})^{2^{k-1}} = a^{\ell/2^{m+1}} = txt^{-1} \in J$, as needed. □

Lemma 6.6. *If J is a non-trivial normal subgroup of B_2 contained in H, then H/J is finite.*

Proof. Since $J \neq 1$, $a^{\ell/2^m} \in J$ for some choice of ℓ and m, and hence $a^\ell \in J$ where $\ell \in \mathbb{Z}$. Since J is invariant by conjugation by t^{-1}, it follows that we can assume also that $a^\ell \in J$, where now ℓ is an odd integer. It follows that B/J can be presented on two generators, which we again denote by a and t satisfying the relations $t^{-1}at = a^2$, $a^\ell = 1$ together with some additional ones. Hence, B_2/J is a quotient of the group K on the generators a, t with defining relations $a^\ell = 1$ and $t^{-1}at = a^2$, which, by Corollary 6.4, is an extension of a cyclic group of order ℓ by an infinite cyclic group. Since B_2/J is a quotient of K, the commutator subgroup of K maps onto the commutator subgroup H/J of B_2/J, and therefore, H/J is finite since the commutator subgroup of B_2/J is finite. □

We are now in a position to complete the proof of Proposition 6.2.

Proof of Proposition 6.2. Suppose that C is a non-abelian subgroup of B_2. Since the commutator subgroup H is abelian, it follows that C is not contained in H. Moreover, the commutator subgroup of C will be a non-trivial subgroup of H. Consequently, C contains, in particular, two elements of the form $t^k a^{\ell/2^m}$, $a^{r/2^s}$, where $k > 0$ and $r \neq 0$. It follows that $J = C \cap H$ is a non-trivial subgroup of H which is invariant under conjugation by t^{-k}. Hence, by Lemma 6.5, J is a normal subgroup of B_2. Since $J \neq 1$, H/J is finite. But B_2/J is an infinite cyclic extension of a finite cyclic group, and therefore, some power of tJ centralizes H/J. It follows that some power of tJ is contained in C/J. Consequently, C/J is of finite index in B/J, which implies that C is of finite index in B_2. □

We can now complete the proof of Theorem 6.1 that the group B_2 satisfies Property R. We wish to show that if C is a subgroup of B_2 and if X is a set of generators for B_2 such that each of the elements of X has a non-trivial power in C, then C is of finite index in B.

We claim first that C is non-abelian. Indeed, observe that B_2 is a U-group, i.e., extraction of roots is unique whenever they exist. So if $x^{-1}y^n x = y^n$, then $(x^{-1}yx)^n = y^n$ from which we deduce that $x^{-1}yx = y$. Consequently, if two non-trivial powers of two elements of B_2 commute, then the elements themselves commute. Alternatively, this can be seen from [26] in which it is proven that B_2 is power commutative, that is, if $[x^m, y^k] = 1$, then $[x, y] = 1$. It follows that C is non-abelian since B_2 is non-abelian.

Since C is non-abelian, we can appeal to Proposition 6.2. Hence, from that result, it follows that C is of finite index in B_2.

From the same arguments as above, it is straightforward to deduce that all of the groups with presentations of the form $\langle a, t;\ t^{-1}at = a^n \rangle$ $(n \geq 2)$ also have this property.

Now in each B_n, the commutator subgroup is not finitely generated. On the other hand, virtually nilpotent groups have the maximum condition, so if B_n were virtually nilpotent, this would not be the case. It follows that each B_n is not virtually nilpotent, completing the proof. □

7 Some Questions

In this section, we give some questions concerning these properties. We noted before that Property S is not preserved under arbitrary free products, the infinite dihedral group provides a counterexample. However, it is open if the factors are torsion-free.

Question 7.1. Is Property S preserved under free products of torsion-free groups satisfying Property S?

Suppose $G = K \star H$ with K and H having Property S. Suppose that A, B are torsion-free subgroups of G with $A \cap B$ having finite index in both A and B. Suppose that A is contained in a conjugate of K or H, say K. Clearly, then $A \cap B$ is also in K. By the Kurosh theorem, B must also be in K since $A \cap B$ has finite index in B, and the result follows in this situation.

Suppose that A is a free product factor of either G or the join $\langle A, B\rangle$, then the result follows as in the free group case.

If A is not a free product factor, then it is not clear what happens.

Question 7.2. Is there an analog of Hall's theorem for general free products of torsion-free groups?

In particular, is there a result of the following form? Let $G = A \star B$. Let $\langle X\rangle$ be a finitely generated torsion-free subgroup not contained in a conjugate of a factor. Then $\langle X\rangle$ is part of a generating system of a free product factor of a subgroup of finite index.

Our methods can be used to prove the subgroup separability of hyperbolic limit groups when the underlying graph is actually a tree. This raises the following question.

Question 7.3. Is there an algebraic characterization of those limit groups whose underlying graph is a tree?

In the hyperbolic case, recall that these all are subgroup separable.

Question 7.4. Is there a purely algebraic or purely function-theoretic proof of Property S for surface groups?

Question 7.5. In a surface group, can Theorem 4.7 be strengthened so that A must be part of a standard generating system?

Recall that a group G has the commensurator property if each non-trivial finitely generated subgroup has finite index in its commensurator.

Question 7.6. Give an example of an infinite finitely generated group G with the commensurator property that is not hyperbolic.

Hyperbolic limit groups satisfy the commensurator condition. In general, we can pose:

Question 7.7. Does a QC-free hyperbolic group satisfy the commensurator condition?

The following has been conjectured to be true by Myasnikov.

Question 7.8. Are hyperbolic limits groups QC-free?

Notice that for torsion-free hyperbolic groups to be QC-free, it is a necessary condition that finitely generated subgroups must be finitely presented. This is known to be true in fully residually free groups and hence in hyperbolic limit groups.

Question 7.9. Classify the finite extensions of Property R groups which still satisfy Property R.

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