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Parafree one-relator groups

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Abstract. Parafree groups are groups that are residually nilpotent and have the property that their quotients by the terms of the lower central series are isomorphic to the corresponding quotients of a free group. We introduce three new families of non-free parafree groups and discuss limitations to a natural procedure for distinguishing these groups from each other.

1 Introduction

1.1 The definition of a parafree group. This paper is concerned with parafree groups, which closely resemble free groups. These groups are generally very hard to distinguish from one another and even from the prototypical parafree groups, the free groups (see [5], [6], [7], [8]).

We begin by introducing some notation in order to be able to recall the definition of a parafree group.

As usual, we denote the conjugate $y^{-1}xy$ of the element x by the element y, where x and y are elements in a group G, by x^y and the commutator $x^{-1}y^{-1}xy$ of x and y by $[x, y]$. We denote iterated commutators $[[[[x_1, x_2], x_3], \ldots], x_n]$ as $[x_1, x_2, \ldots, x_n]$. The lower central series

$$G = \gamma_1(G) \geqslant \gamma_2(G) \geqslant \cdots \geqslant \gamma_n(G) \geqslant \cdots$$

is defined inductively by

$$\gamma_{n+1}(G) = \mathrm{gp}([x, y] \mid x \in \gamma_n(G), y \in G).$$

The group G is termed residually nilpotent if

$$\bigcap_{n=1}^{\infty} \gamma_n(G) = 1$$

Equivalently, G is residually nilpotent if given any non-trivial element $g \in G$, there exists a normal subgroup N of G such that $g \notin N$ with G/N nilpotent. More generally, if $\mathscr{P}$ is a property or class of groups, then G is termed residually $\mathscr{P}$ if, given any non-trivial element $g \in G$, there exists a normal subgroup N of G such that $g \notin N$ with $G/N \in \mathscr{P}$.

We now recall that a group G is *parafree* if

(1) G is residually nilpotent;

and

(2) there exists a free group F such that $G/\gamma_n(G) \cong F/\gamma_n(F)$ for every integer $n \geqslant 1$.

1.2 Three new families of parafree groups. This paper grew out of an attempt to better understand the ideas discussed by Baumslag [5]. The theorems that will be proved here have their origins in that work.

There are now many known families of parafree groups. The primary objective of this paper is to add to this knowledge. The related paper [1] is highly computational and deals with the isomorphism problem for parafree groups. We will briefly discuss that approach later in 1.3.

Before describing the first family of parafree groups, we need to introduce some additional notation. Recall that a non-empty class $\mathscr{V}$ of group is called a *variety of groups* (see H. Neumann [13]) if it is closed under subgroups, epimorphic images and unrestricted direct products. For each such variety $\mathscr{V}$ and each group G, we define

$$V(G) = \bigcap\{N \mid N \text{ normal in } G \text{ and } G/N \in \mathscr{V}\}.$$

The following theorem holds.

Theorem 1. *Let F be a finitely generated free group, freely generated by $a_1, \dots, a_p$ where $p > 1$, and let w be an element in the derived group of F. Furthermore, let m and n be coprime, positive integers and suppose that $a_1^m w$ is not a proper power in F. Then the one-relator group*

$$G = \langle a_1, \dots, a_p, t \mid a_1^m w = t^n \rangle$$

is parafree. Moreover, if $\mathscr{V}$ is any given variety of groups, if $m = 1$ and if $w \in V(F)$, then $G/V(G)$ is free in $\mathscr{V}$. Finally, G is free if and only if either $a_1^m w$ is a primitive element of F or $n = 1$.

The second family of parafree groups consists of what are often referred to as doubles.

Theorem 2. *Let F be a finitely generated free group, freely generated by $a_1, \dots, a_p$, and let w be an element in the derived group of F. If $\overline{F}$ is an isomorphic copy of F, then the amalgamated product*

$$G = \{F * \overline{F} \mid a_1 w = \overline{a_1 w}\}$$

is parafree. Moreover, if $w \in V(F)$, *where* $\mathscr{V}$ *is any given variety of groups, then* $G/V(G)$ *is free in* $\mathscr{V}$. *Finally, G is free if and only if* $a_1 w$ *is a primitive element of F.*

The description of the third family of parafree groups is rather technical: the special cases that follow the statement of the theorem will help to make it more concrete.

We need to prepare for the statement of the theorem in advance. Let E be a free group, freely generated by $s_1, \ldots, s_q$, t and let w be an element in the derived group of E. Suppose that w is cyclically reduced and that t occurs in w. Put

$$s_{i,j} = t^{-j} s_i t^j \quad (i = 1, \ldots, q, j \in \mathbb{Z}).$$

Then the subgroup D of E generated by the $s_{i,j}$ is a normal subgroup of E, freely generated by these elements and E/D is infinite cyclic on tD. Since t occurs with exponent sum 0 in w, it can be uniquely represented by a reduced word w_0 in the generators $s_{i,j}$. Let i be any of the integers $1, \ldots, q$ and suppose that $s_{i,j}$ appears in w_0. We denote by $\mu(i)$ the minimum of all those j for which $s_{i,j}$ appears in w_0 and by $\nu(i)$ the maximum of those j for which $s_{i,j}$ appears in w_0. We then say that w satisfies *the redundancy condition* if there exists an i, say i', such that $\mu(i')$ and $\nu(i')$ are distinct and both $\mu(i')$ and $\nu(i')$ appear once and only once in w_0.

We are now in a position to formulate

Theorem 3. *Let F be the free group on* $a_1, \ldots, a_p$, $s_1, \ldots, s_q$, t *where* $p \geqslant 1$ *and* $q \geqslant 1$. *Furthermore, let E be the subgroup of F generated by* $s_1, \ldots, s_q$, t *and let w be a cyclically reduced word in the derived group of E satisfying the redundancy condition formulated above. Finally, let v be a word in the second derived group of F and suppose that v does not involve* $s_{i'}$. *Then the one-relator group*

$$G = \langle a_1, a_2, \ldots, a_p, s_1, \ldots, s_q, t \mid a_1 = vw \rangle$$

is parafree. Moreover if $v, w \in V(F)$, *where* $\mathscr{V}$ *is any given variety of groups, then* $G/V(G)$ *is free in* $\mathscr{V}$.

There are two special cases of Theorem 3 that we want to draw attention to here.

Corollary 4. *Let*

$$G = \langle a_1, a_2, \ldots, a_p, s_1, t \mid a_1 = v[s_1, t^{e_1}, \ldots, t^{e_l}] \rangle,$$

where v lies in the k-th term of the derived series of the free group on $a_1, \ldots, a_p$, t, *with* $k > 1$. *If* $0 < e_1 \leqslant e_2 \leqslant \cdots \leqslant e_l$, *then G is parafree. Moreover*

$$G/G^{(k)} \cong H/H^{(k)},$$

where H is a free group of rank $p+1$ *and* $G^{(k)}$, $H^{(k)}$ *denote the k-th terms of the derived series of G and H, respectively.*

This family of groups includes many of the groups described in [5] and a great many others as well.

The next corollary provides us with a rather different looking family of parafree groups.

Corollary 5. *Let*

$$G = \langle a_1, a_2, \ldots, a_p, s_1, s_2, \ldots, s_{2q-1}, t \mid a_1 = v[s_1, s_2] \ldots [s_{2q-1}, t] \rangle,$$

where v is a word in the second derived group of the free group on all of the given generators except for s_{2q-1}. Then G is parafree.

The proof Theorem 3 depends on a result of independent interest recorded here as Theorem 4, which was also obtained by Wong [16]:

Theorem 4. *Let G be an extension of a free group N by an infinite cyclic group C. If the abelianization of N, viewed as a module over the integral group ring of C, is free, then G is residually torsion-free nilpotent.*

It is not hard to deduce from Theorem 4 that many of the groups defined by a single relation of the form $a_1 = w[s, t]$ are parafree (see [5]). In particular

$$G = \langle a, s, t \mid a = [s, a][s, t] \rangle$$

is parafree. We concern ourselves with these assertions at the end of Section 4.

1.3 Remarks on the isomorphism problem. The isomorphism problem for one-relator groups seems out of reach at this time. Indeed, there are many problems about one-relator groups that remain open. In particular, there is no known algorithm to decide whether or not a one-relator group is residually nilpotent.

It is clear from the theorems formulated above that there are large collections of parafree one-relator groups. Many of these groups satisfy a sufficiently strong small-cancellation condition to be hyperbolic and so the solution of the isomorphism problem for hyperbolic groups by Sela [14] can be applied to them. However this does not explicitly distinguish them. Indeed, it is often even difficult to distinguish a parafree group from a free group. We have not been able to verify the remark in [5] that the parafree groups described there are not free.

The related paper [1] concerns the isomorphism problem for these families of parafree groups. Although we will not formulate here the results in [1], it seems appropriate to discuss the methods used there.

A one-relator group is free if and only if its defining relator is a primitive element in the ambient free group (Magnus, [3], see also Whitehead [15] and the discussion in Lyndon and Schupp [4]). In [15], Whitehead proved that there is an algorithm which decides whether or not an element in a free group is a primitive. This algorithm appears to involve an exponential number of steps in the number of generators of the free group and so is not easy to invoke if there are many generators. In due course, we will experiment with a genetic version of Whitehead's algorithm, but we content ourselves in [1] with a direct application of Whitehead's algorithm to decide if the groups are non-free. There is another way of distinguishing a given parafree group from another, as well as from free groups, introduced by Lewis and Liriano [11]. In order to explain, let $\mathrm{Hom}(G, T)$ denote the set of all homomorphisms from the group G to the group T. If G is finitely generated and T is finite, $\mathrm{Hom}(G, T)$ is a finite set. Lewis and Liriano compute $\mathrm{Hom}(G, T)$ in the case where G ranges over a parameterized family of one-relator, parafree groups and T over a set of, necessarily solvable, groups of order at most 24. The upshot of their computations is that many of these groups are not isomorphic because they have 'hom' sets of different sizes. Our computations are more demanding in that we choose the target groups T to have larger orders. In particular we make use of alternating groups of degree 5 and 6 and hence the computation of $\mathrm{Hom}(G, T)$ is a daunting task. The results that we have obtained are gathered together in [1].

It is worth noting that any attempt to distinguish one-relator parafree groups from one another using such sets of homomorphisms into finite groups has severe limits. Indeed, if T is any given finite group and if the variety $\mathscr{V}$ involved in the statements of our theorems is chosen appropriately, then for each of the groups G involved

$$|\mathrm{Hom}(G, T)| = |\mathrm{Hom}(E, T)|,$$

where here E is a suitably chosen free group. This means that given any integer n, we can arrange for our groups G to have the same finite images of order at most n as E and therefore the same finite images as each other. Whether a finitely generated non-free, parafree group can have the same finite images as a given free group is a question we have not been able to answer.

1.4 The arrangement of the rest of this paper. The proofs of Theorems 1 and 2 are relatively straightforward. We prove Theorem 1 in Section 2, Theorem 2 in Section 3 and Theorems 3 and 4 in Section 4.

2 Proof of Theorem 1

2.1 Remarks on nilpotent groups. It is well known that if H is a nilpotent group and if X generates H modulo its commutator subgroup, then X generates H. We shall take for granted here various definitions and results about varieties of groups, from, for example H. Neumann [13], in particular, the notion of a free group in a given variety, a free set of generators and the rank of such a free group. The following lemma will be useful.

Lemma 6. *Let H be a finitely generated nilpotent group which is free of rank n in a given variety of nilpotent groups. Then any set of n elements of H which generates H freely generates H.*

This follows from the fact that finitely generated nilpotent groups are hopfian (see Malcev [12]).

If A is a subset of a group B, we denote the subgroup of B generated by A by $\mathrm{gp}(A)$ and the least normal subgroup of B containing A, the normal closure of A in B, by $\mathrm{gp}_B(A)$.

Lemma 7. *Let H be a free nilpotent group of class c and rank n, and let v be an element of H which is not a proper power modulo the derived group of H. Then $H/\mathrm{gp}_H(v)$ is again free nilpotent of rank $n-1$.*

Proof. We choose a finite set $X = Y \cup \{v\}$ of elements containing v which freely generates H, modulo its derived group. It follows that X generates H and therefore, by Lemma 1, X freely generates H. Hence $H/\mathrm{gp}_H(v)$ is free nilpotent of rank $n-1$, as desired.

2.2 Proof of Theorem 1. We will need the following observation.

Lemma 8. *Let H be a free nilpotent group of finite rank n and class c. Let K be a torsion-free nilpotent group containing H such that $K = \mathrm{gp}(H, t)$, where some non-trivial power of t is contained in H. If K can be generated by n elements then K is also free nilpotent, of rank n.*

Proof. Let L be a free nilpotent group of class c of rank n. Since K can be generated by n elements, there exists a homomorphism ϕ from L onto K. Now the torsion-free rank of K (the number of factors in a poly-infinite-cyclic series for K) is at least that of H. Hence, the torsion-free rank of K is the same as that of L. This means that $\ker \phi$ must be trivial.

To prove Theorem 1, we first note that G is residually torsion-free nilpotent by [7]. Now, G can be generated modulo its derived group by p elements, since m and n are relatively prime. Therefore $G/\gamma_{c+1}(G)$ can be generated by p elements. We next observe that given any torsion-free nilpotent group H, a positive integer j and an element v in H, there exists, by a theorem of Malcev [12], a torsion-free nilpotent group K containing H in which v has a jth root. If we now take H to be a free nilpotent group of class c and rank p freely generated by $x_1, \dots, x_p$ and adjoin for $x_1^m w(x_1, \dots, x_p)$ an nth root r in a torsion-free nilpotent group then, by Lemma 8, $K = \mathrm{gp}(H, r)$ is free nilpotent of rank p and class c. The group K is clearly a quotient of $G/\gamma_{c+1}(G)$. But $G/\gamma_{c+1}(G)$ can be generated by p elements. Hence $G/\gamma_{c+1}(G)$ is a free nilpotent group of rank p and class c.

In order to complete the proof of Theorem 1 it remains only to verify the following

Lemma 9. *The groups G defined in Theorem 1 are free if and only if aw is a primitive element in the free group F.*

The proof of Lemma 4 is straightforward. Clearly G is free if aw is a primitive element. If aw is not a primitive element, then the amalgamated product

$$P = (F/\mathrm{gp}_F(aw)) * (\mathrm{gp}(t)/\mathrm{gp}(t^n))$$

is a quotient of G. Now it follows from a theorem of Magnus [2] that a one-relator group presented on p generators and one relator can be generated by $p-1$ generators if and only it is free. Thus, the relator involved is part of a free basis of the ambient free group and is thus primitive. It follows from Gruschko's Theorem [9] that P cannot be generated by fewer than $p+1$ generators. Therefore G is not free.

3 Proof of Theorem 2

A double of a free group, where the amalgamated subgroup is cyclic, is residually free [6], provided only that a generator of the subgroup is not a proper power. So G is residually free and hence residually torsion-free nilpotent. Now $aw(\overline{aw})^{-1}$ is not a proper power modulo the derived group of the free group on $X \cup \overline{X}$. It follows from Lemma 6 that the quotients of G modulo the terms of its lower central series are free nilpotent. This completes the proof that G is parafree. Now if aw is not a primitive element in the free group F, then $F/\mathrm{gp}_F(aw)$ is generated by p elements, but not by fewer. Hence the free product $F/\mathrm{gp}_F(aw) * \overline{F}/\mathrm{gp}_{\overline{F}}(\overline{aw})$ requires $2p$ generators. It follows that G cannot be generated by fewer than $2p$ generators. Hence G is not free.

4 Proof of Theorem 3

4.1 First step in the proof. We term the groups defined in Theorem 3, groups of Type III. The first step in the proof of Theorem 3 is the following

Lemma 10. *Groups of Type III are extensions of free groups by infinite cyclic groups; that is, free by infinite cyclic.*

Each of the groups defined in Theorem 3 is defined by a single relator

$$r = a_1^{-1} v w.$$

Then t occurs in r with exponent sum zero and so we can rewrite r in terms of the generators

$$a_{i,j} = t^{-j} a_i t^j, \quad s_{i,j} = t^{-j} s_i t^j$$

with j ranging over the set of all integers. It follows from Magnus' basic breakdown of a one-relator group [2] that

$$N_0 = \mathrm{gp}(a_{1,m(1)}, \dots, a_{1,M(1)}, \dots, a_{p,m(p)}, \dots, a_{p,M(p)}, s_{1,\mu(1)}, \dots, s_{1,\nu(1)} \dots, s_{q,\mu(q)}, \dots, s_{q,\nu(q)})$$

is defined by the single relator r_0 obtained by rewriting r in terms of the generators $a_{i,j}, s_{i,j}$; here $m(i)$ and $M(i)$ are the maximum subscripts that arise from the a_i and the $\mu(i)$ and $\nu(i)$ are the subscripts obtained from w in the manner detailed in the definition of the redundancy condition. Thus, again adopting the notation developed previously, $\mu(i')$ and $\nu(i')$ are distinct and occur once and only once in r_0. Since $s_{i'}$ does not occur in v, it follows that both $s_{i',\mu(i')}$ and $s_{i',\nu(i')}$ occur once and only once in r_0. It follows that we can express $s_{i',\nu(i')}$ in terms of the remaining generators and hence N_0 is free. Similarly if we adjoin tN_0t^{-1} to N_0, we can express $s_{i',\mu(i')-1}$ in terms of the generators we have obtained for N_0 and the generators for tN_0t^{-1} with $s_{i',\nu-1}$ now omitted. It follows that $\mathrm{gp}(N_0, tN_0t^{-1})$ is again free. Hence the normal closure N of N_0, which is obtained from N_0 by successively adjoining the conjugates $t^{-i}N_0t^i$ of N_0, $i = 1, -1, 2, -2, \ldots$, one at a time, is free. Since G/N is infinite cyclic, this completes the proof.

4.2 Proof of Theorem 4. The main step in the proof of Theorem 3 is the proof that groups of type III are residually nilpotent. It turns out that this is a consequence of Theorem 4, the proof of which we concern ourselves with next. To this end, let N be a normal subgroup of a group G and let G/N be infinite cyclic with generator tN. Let $C = \mathrm{gp}(t)$ and let Λ be the integral group ring of C. Then N_{ab} can be viewed as a Λ-module with t acting on Λ by conjugation. The assumption is that this Λ-module is free. Our objective is to prove that G is residually torsion-free nilpotent.

Let $\{x_i\gamma_2(N) \mid i \in I\}$ be a free basis for N_{ab}. Then the set

$$X = \{x_{i,j} = t^{-j}x_it^j \mid i \in I, j \in \mathbb{Z}\}$$

freely generates, modulo $\gamma_2(N)$, a free abelian group. Hence X freely generates, modulo $\gamma_{c+1}(G)$, a free nilpotent group of class c. In order to complete the proof of Theorem 4, it suffices to prove the following

Lemma 11. *Let Z be a free nilpotent group of class c, freely generated by the generators*

$$z_{i,j} \quad (i \in I, j \in \mathbb{Z}).$$

Furthermore, let W be the semi-direct product of Z and an infinite cyclic group on z, where z acts on Z as follows:

$$z^{-1}z_{i,j}z = z_{i,j+1} \quad (i \in I, j \in \mathbb{Z}).$$

Then W is residually torsion-free nilpotent.

Proof. Let k be a positive integer and let $D(k)$ be the free nilpotent group of class c, freely generated by the elements

$$y_{i,j} \quad (i \in I, j = 1, \ldots, k).$$

Define $E(k)$ to be the semi-direct product of $D(k)$ by the infinite cyclic group on y, where y acts on $D(k)$ as follows:

$$y^{-1}y_{i,j}y = y_{i,j}y_{i,j+1} \quad (i \in I, 1, j = 1, \dots, k-1),$$
$$y^{-1}y_{i,k}y = y_{i,k}.$$

It is easy to see that $\gamma_{k+1}(E(k)) \leqslant \gamma_2(D(k))$. By a theorem of P. Hall [10], $E(k)$ is nilpotent. It is clearly torsion-free. Now the elements

$$y^{-l}y_{i,1}y^l \quad (l = 0, 1, \dots, k-1, i \in I)$$

freely generate $D(k)$. There is a canonical homomorphism of W onto $E(k)$, with z mapping onto y and the $z_{i,1}$ mapping onto $y_{i,1}$. If k is chosen sufficiently large, then any finite set of the given free generators of Z will map onto a set of free generators of $D(k)$. It follows that if w is any non-trivial element of W, then by choosing k sufficiently large, we can arrange that its image in $E(k)$ is non-trivial. So W is residually torsion-free nilpotent.

We can now prove that groups of Type III are residually nilpotent.

Lemma 12. *Let G be a group of Type III*:

$$G = \langle a_1, a_2, \dots, a_p, s_1, \dots, s_q, t \mid a_1 = vw \rangle.$$

Furthermore, let N be as above, i.e., the normal closure of the the set of all of the generators of G excluding t. Finally, let Λ be the integral group ring of $C = \mathrm{gp}(t)$. Then N_{ab}, viewed as a module over Λ, with t acting on N_2, is free.

Proof. The Λ-module $N_{\mathrm{ab}} = N/\gamma_2(N)$ is generated by the images of the elements $a_1, \dots, a_p$, $s_1, \dots, s_q$ modulo $\gamma_2(N)$ subject to the single module relation $a_1\gamma_2(N) = vw\gamma_2(N)$. Since $v \in \gamma_2(F)$, $v \in \gamma_2(N)$ and since t occurs with exponent sum 0 in w, it follows that this relation can be rewritten in the form $a_1\gamma_2(N) = w'\gamma_2(N)$, where w' is a word in the generators $s_1, \dots, s_q$. It follows that M is a free Λ-module with free basis the images, modulo $\gamma_2(N)$, of the elements $a_2, \dots, a_p$, $s_1, \dots, s_q$.

4.3 Third step in the proof of Theorem 3. It follows from Theorem 4 and Lemma 12 that G is residually nilpotent. Lemma 7 then applies and it shows that $G/\gamma_{c+1}(G)$ is free nilpotent of class c for every c. This completes the proof of Theorem 3.

4.4 One corollary of Theorem 4.

Corollary 13. *The group*

$$G = \langle a, s, t \mid a = [t, a][s, t] \rangle$$

is parafree.

Proof. It follows from Magnus' basic breakdown [2] of a one-relator group that the normal closure N of $\{a, s\}$ is a free group, freely generated by $t^{-i}at^{i}$ $(i \in \mathbb{Z})$, together with s. Notice that

$$a = t^{-1}a^{-1}tas^{-1}t^{-1}st.$$

This means that N_{ab}, viewed as a module over the integral group ring Λ of the infinite cyclic group generated by t, is generated by $a\gamma_2(N)$ and $s\gamma_2(N)$ and defined in terms of these generators by the single relation

$$a\gamma_2(N) = at^{-1}a^{-1}ts^{-1}t^{-1}st\gamma_2(N).$$

It follows that

$$t^{-1}at\gamma_2(N) = s^{-1}t^{-1}st\gamma_2(N),$$

and therefore the Λ-module N_{ab} is free on $s\gamma_2(N)$. So Theorem 4 applies, and hence G is residually nilpotent. On applying Lemma 7, we find that G is indeed parafree.

It is clear that the above corollary can be formulated in rather more general terms; we leave such a formulation to the reader.

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