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Some Recognizable Properties of Solvable Groups

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§1. Introduction

In the variety of all groups the well-known theorem of Adian [1, 2] and Rabin [8] shows that most group-theoretic properties can not be effectively recognized. We will see the situation is quite different if one assumes the variety is $\mathfrak{A}^k$, the variety of k-step solvable groups.

A group G is said to be *finitely presented in* $\mathfrak{A}^k$ if G is generated by $x_1, \ldots, x_n (n<\infty)$ subject to the defining relators $A_1, \ldots, A_p (p<\infty)$ plus the laws of $\mathfrak{A}^k$. We use the expression

$$\langle k \langle x_1, \ldots, x_n; A_1, \ldots, A_p \rangle\rangle \tag{1}$$

to denote a presentation of such groups and we put $FP\mathfrak{A}^k$ as the class of such groups ($k \geqq 1$).

In this note we prove the

Theorem 3.1. *There is an algorithm which when given a presentation of type* (1) *decides if the group presented is polycylic, and if so, produces a finite presentation for the group.*

This result has the immediate

Corollary 3.2. *The following properties of a group are effectively recognizable from a presentation of type* (1):

a) *polycyclic,*
b) *supersolvable,*
c) *nilpotent,*
d) *abelian,*
e) *cyclic,*
f) *finite,*
g) *trivial.*

As an immediate consequence of Theorem 3.1 we have the

Theorem 3.3. *There is an algorithm which when given an arbitrary finitely presented solvable group G, decides if G has any of properties* (a)–(g) *above.*

Remark. Properties (d)–(g) are long known to be decidable for the class of groups under consideration, on virtually trivial grounds, and are included here primarily for the sake of completeness. Property (c) nilpotent was discussed by the second author in [5], in answer to Problem 8 of Remeslennikov and Romanovskii [7]. However, the method employed in [5] failed to generalize to cases (a) and (b) considered here.

The discussion and proofs will be based on our [3] and [4]. It will be assumed the reader has a copy of these papers at hand, from which all undefined terminology has been taken.

§2. Extension of a Result of [4]

The following lemma, which extends Theorem 4.1 of [4] will be needed in the proof of the theorem. In addition we record a related theorem of independent interest.

Lemma 2.1. *Let P be a polycyclic group with integral group ring ZP. Then there is an algorithm to decide whether or not an arbitrary finitely presented ZP-module M is finitely generated as an abelian group.*

Proof. The proof is by induction on the length of any polycyclic series for P. We may assume inductively that P has a normal subgroup H such that P/H is infinite cyclic, else we consider M as a module over ZH. By Theorem 4.3 of [3] we may effectively find a finite presentation for H from a given finite presentation for P.

Assume $M \cong F/N$ where F is the free ZP-module freely generated by $\xi_1, \ldots, \xi_q (q<\infty)$ and N is the ZP-module generated by $v_1, \ldots, v_r$. Let $H \lhd P$, such that we have the transversals

$$P=\bigcup x^i H=\bigcup H x^i. \tag{2}$$

Put

$$F_z=\operatorname{mod}_{ZH}(\xi_i x^j\colon\ 0\leqq j\leqq z,\ 1\leqq i\leqq q)$$

and let d be an integer larger than the degrees of any of the $v_i (1\leqq i\leqq r)$. By Lemma 2.2 and the proof of Theorem 2.12 of [4] we can effectively find a finite set Σ_z of ZH-module generators for $F_z \cap N$, and for $z=d$, Σ_d generates N as a ZP-module. Furthermore we can find a finite presentation for $F_d \cap N$ and for

$$F_d/F_d \cap N, \tag{3}$$

which by the induction hypothesis we can assume is finitely generated as an abelian group else F/N is not. Put F_0 as the ZH-submodule of F generated by $\xi_1, \ldots, \xi_q$ and let K_z be the ZH-submodule of F_0 generated by the leading coefficients of x^z in the set Σ_z of generators for $F_z \cap N$, $(0\leqq z\leqq d)$. Finally, let

$$K=K_0+K_1+\ldots+K_d. \tag{4}$$

Similarly for $0\leqq z\leqq d$ put

$$F'_z=\operatorname{mod}_{ZH}(\xi_i x^j\colon\ z\leqq j\leqq d,\ 1\leqq i\leqq q).$$

Again we can effectively find a finite set Γ_z of ZH-module generators for $F'_z \cap N$. Let L_z be the ZH-submodule of F_0 generated by the trailing coefficients of x^z in the set Γ_z of generators for $F'_z \cap N$, $(0\leqq z\leqq d)$ and finally put $L=L_0+L_1+\ldots+L_d$.

We claim F/N is finitely generated as an abelian group if and only if $K=F_0=L$.

$(\Rightarrow)$ Assume $K=F_0=L$ and let $f\in F$, say

$$f=a_s x^s+\ldots+a_t x^t. \tag{5}$$

If $t>d$ then $a_t\in K(=F_0)$ and so for some $g\in F_d\cap N$

$$f-g x^{t-u}=b_v x^v+\ldots+a_u x^u\equiv f(\operatorname{mod} N), \tag{6}$$

where $\min(s,0)\leqq v$ and $0\leqq u<t$. Furthermore, if in (5) $s<0$ then $a_s\in L(=F_0)$ and there is $g\in F_d\cap N$ with

$$f-g x^{-s}=b_v x^v+\ldots+a_u x^u\equiv f(\operatorname{mod} N), \tag{7}$$

where $s<v\leqq 0$ and $u\leqq\max(t,d)$. Thus in finitely many rewritings (6) and (7) we get $f\equiv$(an element of F_d) (mod N). It follows that $F_d+N=F$ and so by a standard isomorphism theorem

$$F/N\cong F_d/F_d\cap N,$$

and F/N is finitely generated as an abelian group.

$(\Leftarrow)$ Observe that by the construction of K and L any $f\in N$ has leading coefficient in K and trailing coefficient in L. In case $K\neq F_0$, pick a fixed element $b\in F_0-K$. Then $bx^i\notin N$ for any $i\geqq 0$. Moreover bx^i is not congruent modulo N to any Z-linear combination of $\{b, bx, bx^2, \ldots, bx^{i-1}\}$. For such a congruence would yield an element of N with leading coefficient $b\notin K$ contradicting our previous observation. Hence in this case the abelian subgroup of F/N generated by $\{bx^i, i\geqq 0\}$ is not finitely generated.

Similarly in case $L\neq F_0$, pick a fixed $b\in F_0-L$. The analagous argument shows that the abelian subgroup of F/N generated by $\{bx^i, \leqq 0\}$ is not finitely generated.

By Lemma 2.2 of [4] there is an algorithm to decide if $K=F_0=L$. This completes the proof. □

Of independent interest is the next result, motivated by Lemma 2.1 and Theorem 2.12 of [4].

Theorem 2.2. *Let G be a finitely generated nilpotent-by-polycyclic group satisfying max-n. Then G has solvable word problem (and, hence, is recursively presented).*

Proof. Let $1 \to N \to G \to P \to 1$ be short exact, N nilpotent, G finitely generated and P polycyclic. Put $N = \gamma^1(N) \trianglerighteq \gamma^2(N) \trianglerighteq \ldots \trianglerighteq \gamma^c(N) \trianglerighteq \gamma^{c+1}(N) = 1$ as the lower central series of N. Then $M_i = \gamma^i(N)/\gamma^{i+1}(N)$ is a finitely generated right ZP-module $(1 \leqq i \leqq c)$ and, since ZP is (right) submodule computable (see [4], Theorem 2.12), M_i is in fact effectively finitely presentable and has solvable word problem. Thus, to solve the word problem in G we can assume the word in question lies in N, since $P = G/N$ has solvable word problem. At this point one successively goes down the series of modules M_i to decide if the word is 1 in G. This completes the proof.

Remark. In [6] it was proposed that the exact boundary for unsolvability of the word problem in 3-step solvable groups would involve the variety center-by-metabelian. Theorem 2.2, contributes some information on this conjecture.

§3. Recognizing Polycyclic Groups

In this section we prove the main theorem, which generalizes Corollary 4.2 of [4].

Theorem 3.1. *There is an algorithm which decides whether a group* $G \in FP\mathfrak{A}^k$ *given by a presentation of type* (1) *is polycyclic, and if so, produces a finite presentation for* G.

Proof. Let $G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq \ldots \trianglerighteq G^{(k-1)} \trianglerighteq G^{(k)} = 1$ be the derived series for G. Applying the induction hypothesis to $\langle k-1 \langle x_1, \ldots, x_n;\ A_1, \ldots, A_p \rangle\rangle$ which presents $G/G^{(k-1)}$, we may assume $P = G/G^{(k-1)}$ is polycyclic and that we have obtained a finite presentation

$$P = \mathrm{gp}(a_1, \ldots, a_r; R_1, \ldots, R_s). \tag{8}$$

The abelian normal subgroup $G^{(k-1)}$ of G can be regarded as a (right) ZP-module, denoted M, under the action

$$g^v = \prod_{i=1}^{m} (u_i^{-1} g u_i)^{e_i}$$

where $g \in M$, $v \in ZP$, $v = \sum_{i=1}^{m} (u_i G^{(k-1)}) e_i$, and $e_i \in Z$. Since M is (right) Noetherian as a ZP-module, and hence finitely presentable, we show how to effectively present M.

First, note that $R_1, \ldots, R_s$ are a set of ZP-module generators for M. We introduce the symbols $\hat{R}_1, \ldots, \hat{R}_s$ to represent these generators.

Next, let $\rho_1, \rho_2, \ldots, \rho_t, \ldots$ be a recursive enumeration of the relators in G. Since $\rho = 1$ in G implies $\rho G^{(k-1)} = 1$ in P we obtain the relation

$$\rho = \prod_{i=1}^{m} u_i^{-1} R_{\alpha(i)}^{e_i} u_i, \tag{9}$$

holding in the free group on basis $x_1, \ldots, x_n$ for some $u_i \in P, e_i \in Z$ and $1 \leqq \alpha(i) \leqq s$. This gives rise to a module relator (written additively)

$$\hat{\rho} = \sum_{i=1}^{m} \hat{R}_{\alpha(i)}(u_i e_i). \tag{10}$$

Hence, let M_t be the finitely presented ZP-module with generators $\hat{R}_1, \ldots, \hat{R}_s$ and defining relators $\hat{\rho}_1, \ldots, \hat{\rho}_t$, corresponding to the first t relators of G in the enumeration, plus certain other relators – finitely many and the same for all t – to be discussed below.

3.1. We next associate to the module M_t a group G_t as follows: Let Π_t be the following group presentation:
Generators of Π_t: $x_1, \ldots, x_n, \hat{R}_1, \ldots, \hat{R}_s$.
Defining relations of Π_t: $R_i = \hat{R}_i (1 \leqq i \leqq s)$ (11)

$$[\hat{R}_i^u, \hat{R}_j^v] = 1, \tag{12}$$

where u, v are arbitrary words on $x_1, \ldots, x_n$ and $1 \leqq i < j \leqq s$,

$$\rho_1 = 1, \ldots, \rho_t = 1.$$

Digression. We put $G_t = \mathrm{gp}(\Pi_t)$. Now M_t as presented thus far may not be embeddable in G_t as the ZP-module on $\hat{R}_1, \ldots, \hat{R}_s$ because the Eqs. (11) and (12) may imply some ZP-module relations among the $\hat{R}_i$ which do not hold in M_t. Thus, a quotient of M_t is embeddable in G_t. By Lemma 3.0 of [4] the finitely many additional module relators needed to assure M_t is embeddable in G_t are the same for all t and can be effectively found. We assume, henceforth, that the presentation obtained for M_t contains these "additional" relators.

3.2. The group G_t has the following properties:
(i) G_t may be mapped epimorphically on G under the obvious map.
(ii) $G_t \in \mathfrak{A}^k$.
(iii) G_t has solvable word problem.
Only (iii) needs discussic

By a theorem of P. Hall f.g. abelian-by-polycyclic groups satisfy max-n. Hence, Theorem 2.2 applies.

3.3. We can now conclude the proof of the theorem. The Noetherian property of M requires that there be a t_0 such that $M_{t_0} = M_{t_0+1} = \ldots = M$. This will occur when $G_{t_0} \cong G$. The map $G_t \to G$ is invertible if and only if the relators $A_1, \ldots, A_p$ are equal to 1 in G_t because the varietal relators of G are automatically valid in G_t by (ii). Using the solution to the word problem in G_t we may effectively check this for successive t's, and thus eventually obtain a finite presentation for M effectively.

As G is polycyclic if and only if M is finitely generated as an abelian group, we see by Lemma 2.1 we may effectively decide this. If so, to present G finitely we observe that in the presentation Π_{t_0}, the only relators (12) needed are those such that $\hat{R}_i^u$ or $\hat{R}_j^v$ are generators of the finitely generated abelian group, $M = G^{(k-1)}$. Since the rank ℓ of $G^{(k-1)}$ can be effectively found, we use the solution to the generalized word problem (=membership problem for subgroups) for G to find ℓ distinct elements $\hat{R}_i^u$ which generate $G^{(k-1)}$. The proof is now complete. □

Corollary 3.2. *The following properties of a group may be effectively recognized from a presentation of type* (1).

a) *polycyclic,*
b) *supersolvable,*
c) *nilpotent,*
d) *abelian,*
e) *cyclic,*
f) *finite,*
g) *trivial.*

Proof. For, in order for any of (a)–(g) to hold the group presented must be polycyclic and by Theorem 4.3 of [3] we may effectively obtain from a given finite presentation of the group, a finite presentation from which we can effectively determine if the property in question holds. □

We observe that the following algorithm holds in $FP \cap \mathfrak{A}^\infty$, where FP is the class of finitely presented groups and $\mathfrak{A}^\infty = \bigcup_{k \geq 1} \mathfrak{A}^k$.

Theorem 3.3. *We may effectively recognize properties* (a)–(g) *for groups in* $FP \cap \mathfrak{A}^\infty$. *That is, there is an algorithm which, when given a finite presentation* Π *of a solvable group, decides if* gp(Π) *has properties* (a)–(g).

Proof. Clearly it suffices to recognize property (a) polycyclic. If $\Pi = \langle b_1, \ldots, b_m; B_1, \ldots, B_n \rangle$ is a finite presentation denote by $\langle k, \Pi \rangle$ the presentation

$$\langle k \langle b_1, \ldots, b; B_1, \ldots, B_n \rangle \rangle \tag{13}$$

of type (1). We set in process two infinite searches. In the first we apply successively the algorithm of Theorem 3.1 to $\langle 1, \Pi \rangle, \langle 2, \Pi \rangle, \ldots$. In the second one Tietze transforms Π repeatedly. If gp(Π) is not polycyclic a k will be found such that $\langle k, \Pi \rangle$ is found not to be polycyclic. If gp(Π) is polycyclic, an honest polycyclic presentation (cf. [3]) for gp(Π) will appear in the second search. By Theorem 4.3 of [3] we may effectively recognize when this occurs. Since one or the other of the two searches must terminate in finitely many steps the corollary is proved. □

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