International Journal of Algebra and Computation
Vol. 16, No. 3 (2006) 475–491


# SUBSPACES, FREE SUBGROUPS AND A THEOREM OF STALLINGS

GILBERT BAUMSLAG

*Department of Mathematics, City College of New York*
*New York, NY 10031, USA*

Received 13 May 2004
Revised 15 February 2005

Communicated by J. Meakin

There is a simple group-theoretic formula for the second integral homology group of a group. This is an abelian group and there is an analogous formula for another abelian group, which involves a normal subgroup $N$ of a torsion-free nilpotent group $G$. Properties of this abelian group translate into properties of $G/N$. This approach allows one to give a simple purely group-theoretic proof of an old theorem of J. R. Stallings, namely that if $\Gamma$ is a group, if $H_1(G, \mathbb{Z})$ is free abelian and if $H_2(G, \mathbb{Z}) = 0$, then any subset $Y$ of $G$ which is independent modulo the derived group of $G$, freely generates a free group. The ideas used admit to considerable generalization, yielding in particular, proofs of a number of theorems of U. Stammbach.

*Keywords*: Nilpotent groups; homology group; groups with unique roots.

Mathematics Subject Classification: Primary 20F18

## 1. Stallings' Theorem

Almost 40 years ago, Stallings [12] proved the following:

**Theorem.** *Suppose that the abelianization of the group $\Gamma$ is free and that $Y$ is a subset of $\Gamma$ which freely generates modulo $[\Gamma, \Gamma]$ a free abelian group. If $H_2(\Gamma, \mathbb{Z}) = 0$, where $H_2(\Gamma, \mathbb{Z})$ denotes the second integral homology group of $\Gamma$, then $gp(Y)$ is a free group, freely generated by $Y$.*

I noticed at the time that there was a simple group-theoretic proof of Stallings' theorem, which amounted to little more than the observation that any subspace of a vector space has a complement or slightly more generally, that an abelian group which is an extension of an abelian group by a free abelian groups splits. Stallings' proof was homological and was subsequently considerably and elegantly generalized by Stammbach [13]. I have recently re-examined my original argument and this note grew out of this examination. Both Stallings' theorem and many of Stammbach's results are consequences of either of my two main results here, Theorem A1 and

Theorem B1, which will be formulated below. Each of them can be deduced from the other, with only a little work. Theorem A1 applies directly to everyday groups whereas Theorem B1 applies to $\mathbb{Q}$-groups, i.e., groups in which extraction of $n$th-roots is uniquely possible for every choice of the positive integer $n$ and where the basic idea is more clearly understood. The connection between these two theorems comes about because of a theorem of Malčev [11] which allows us to carry over properties of torsion-free nilpotent groups to those of nilpotent $\mathbb{Q}$-groups and conversely. This is essentially the content of Theorem C, which is formulated in Sec. 7.

## 2. The $\mathcal{H}$-Condition

If $X$ is a subset of $G$, then we denote the subgroup of $G$ generated by $X$ by $gp(X)$. The normal closure of the subset $X$ of $G$ is denoted by $gp_G(X)$. The commutator $x^{-1}y^{-1}xy$ of the elements $x$ and $y$ of $G$ is denoted by $[x, y]$ and if $J$ and $K$ are subgroups of $G$, then we define $[J, K] = gp([x, y] \mid x \in J, y \in K)$. We refer to $G_{ab} = G/[G, G]$ as the abelianization of $G$.

There is a group-theoretic description of the second integral homology group $H_2(G, \mathbb{Z})$ of a group $\Gamma$. If $\Gamma \cong F/N$, where $F$ is any given free group, then $H_2(\Gamma, \mathbb{Z}) \cong ([F, F] \cap N)/[F, N]$. So $H_2(\Gamma, \mathbb{Z}) = 0$ translates into $[F, F] \cap N = [F, N]$. This leads to the following:

**Definition 1.** The normal subgroup $N$ of the group $G$ satisfies the $\mathcal{H}$-condition if

$$[G, G] \cap N = [G, N].$$

It turns out that if $N$ is a normal subgroup of a group $G$ satisfying the $\mathcal{H}$-condition and some additional ones, then $\Gamma = G/N$ can be likened to a complement $C$ of $N$ in $G$. It follows, again given the right assumptions, that the quotients of $\Gamma$ by the terms of its lower central series coincide with the corresponding quotients of this subgroup $C$ of $G$. In the event that $G$ is free in some variety $\mathcal{V}$ of groups and if $\Gamma_{ab}$ is free abelian, then $\Gamma$ has the same lower central factors as a free group in $\mathcal{V}$. This allows us, on appealing to variants of a theorem of Magnus [9], to prove not only Stallings' theorem, but various varietal analogs of his theorem as well.

## 3. The Proof of Theorem A1 and Some of Its Consequences

### 3.1. *Theorem A1*

**Theorem A1.** *Let $G$ be a torsion-free nilpotent group, let $N$ be a normal subgroup of $G$ and let $\Gamma = G/N$. Suppose that $N$ satisfies the $\mathcal{H}$-condition, i.e.,*

$$[G, G] \cap N = [G, N]$$

*and that both $G_{ab}$ and $\Gamma_{ab}$ are free abelian. Then there exists a subset $X = Y \cup Z$ of $G$ such that $X$ freely generates $G$ modulo $[G, G]$, $Z$ freely generates $G$ modulo $[G, G]N$ and $gp_G(Y) = N$.*

In order to demonstrate the relevance of Theorem A1 at this stage, we remark that if we assume the notation and hypothesis of Theorem A1 and if we assume that $G$

is a nilpotent group which is free in some variety of groups, then $G$ splits over $N$ and $\Gamma$ is free in the given variety (see Corollary 4, below).

In order to prove Theorem A1 we shall first need to prove that the $\mathcal{H}$-condition is sometimes preserved under homomorphic images. This is the content of the following simple lemma, which plays an important part in the proof of Theorem A1.

**Lemma 2.** *Let $G$ be a group and let $N$ be a normal subgroup of $G$ satisfying the $\mathcal{H}$-condition. If $M$ is any normal subgroup of $G$ which is contained in the derived group of $G$, then the subgroup $NM/M$ of $G/M$ satisfies the $\mathcal{H}$-condition.*

**Proof.** It suffices to note that

$$\begin{aligned}[G/M, G/M] \cap (NM/M) &= ([G,G]/M) \cap (NM/M) = ([G,G] \cap NM)/M \\ &= ([G,G] \cap N)M/M = [G,N]M/M = [G/M, NM/M],\end{aligned}$$

as desired. □

We are now in a position to prove Theorem A1.

**Proof.** The canonical homomorphism from $G_{ab}$ onto $G/[G,G]N \cong \Gamma_{ab}$ maps the free abelian group $G_{ab}$ onto the free abelian group $G/[G,G]N$. Hence the extension

$$1 \to N[G,G]/[G,G] \to G_{ab} \to G/N[G,G] \to 1$$

splits:

$$G/[G,G] = N[G,G]/[G,G] \times E/[G,G],$$

where $E$ is a subgroup of $G$ containing $[G,G]$. Since $G/[G,G]$ is free abelian, we can find a subset $Y$ of $N$ and a subset $Z$ of $E$ such that $\{y[G,G] \mid y \in Y\}$ freely generates the free abelian group $N[G,G]/[G,G]$ and $\{z[G,G] \mid z \in Z\}$ freely generates the free abelian group $E/[G,G]$. So $X = Y \cup Z$ freely generates $G$ modulo $[G,G]$. Consequently, since the Frattini subgroup of a nilpotent group contains the derived group, it follows also that $X$ generates $G$.

Now let $K$ be the normal closure of $Y$ in $G$. We claim that $K = N$. Clearly $K \leq N$. Our objective is to prove the reverse inequality. The proof is by induction on the class $c$ of $G$. If $c = 1$, this is obvious. Assume that $c > 1$. By Lemma 2, the subgroup $N\gamma_c(G)/\gamma_c(G)$ of $G/\gamma_c(G)$ satisfies the $\mathcal{H}$-condition. Moreover $(G/\gamma_c(G))/([G,G]/\gamma_c(G))$ is free abelian and so too is the abelianization $G/[G,G]N$ of

$$(G/\gamma_c(G))/N\gamma_c(G)/\gamma_c(G) \cong G/N\gamma_c(G).$$

So we find inductively that $N \leq K\gamma_c(G)$. It follows that

$$[N,G] \leq [K\gamma_c(G), G] = [K,G].$$

Let $a \in N$. Then $a = bb'$, where $b \in K$ and $b' \in \gamma_c(G)$. Hence $b' \in N \cap [G,G]$. Since $N \cap [G,G] = [N,G]$ and $[N,G] \leq [K\gamma_c(G), G] = [K,G]$, $b' \in K$. Consequently $a \in K$, which implies that $N \leq K$. This completes the proof. □

In order to make use of Theorem A1 we will need the following:

**Lemma 3.** *Let $G$ be a group which is free in a variety of nilpotent groups, freely generated by the set $X = \{x_\ell \mid \ell \in L\}$. Then any set $X'$ of elements of $G$ which freely generates $G$ modulo $[G,G]$, freely generates $G$.*

**Proof.** Since $X$ and $X'$ have the same cardinality, we can assume that $X' = \{x'_\ell \mid \ell \in L\}$. Moreover as $X'$ generates $G$ modulo $[G,G]$, $X'$ generates $G$. Therefore the endomorphism $\phi$ of $G$ defined by

$$\phi : x_\ell \mapsto x'_\ell \quad (\ell \in L)$$

is onto. It remains to show that $\phi$ is one-to-one. Suppose then that $w = w(x_1, \ldots, x_m) \neq 1$ in $G$. We have to prove that $w' = w(x'_1, \ldots, x'_m) \neq 1$. Now taking for granted the appropriate notation, we can find $n \geq m$ so that $x'_\ell \in G' = gp(x_1, \ldots, x_n)$ for $\ell = 1, \ldots, m$. Notice that $G'$ is again a free group in the same variety as $G$. Since $X'$ freely generates $G$ modulo $[G,G]$, any subset of $X'$ generates a direct factor of $G$ modulo $[G,G]$. So $x'_1, \ldots, x'_m$ generates a direct factor of $G'$ modulo $[G',G']$ and hence we can extend the set $\{x'_1, \ldots, x'_m\}$ to a set of elements $\{x'_1, \ldots, x'_m, y_1, \ldots, y_{n-m}\}$ of $G'$ which freely generates $G'$ modulo $[G',G']$. This set then generates $G'$ and so the homomorphism $\gamma$ from $G'$ to $G'$ mapping each $x_\ell$ to $x'_\ell$ for $\ell = 1, \ldots, m$ and $x_{m+k}$ to $y_k$ for $k = 1, \ldots, n-m$ is an epimorphism. Now finitely generated nilpotent groups are hopfian and consequently $\gamma$ is actually an automorphism of $G'$. Since $w\gamma = w'$, this implies that $w' \neq 1$ as desired. □

It is worth noting that we have proved that if $G$ is a group that is free in some variety of nilpotent groups and if $\phi$ is an epimorphism of $G$ which induces an automorphism of $G/[G,G]$, then $\phi$ is an automorphism of $G$. This is not true for nilpotent groups as a whole. The interested reader might wish to construct a necessarily infinitely generated counter-example.

**Corollary 4.** *Let $G$ be a group, free in some variety $\mathcal{V}$ of nilpotent groups and let $N$ be a normal subgroup of $G$ satisfying the $\mathcal{H}$-condition. If the abelianizations of both $G$ and $\Gamma = G/N$ are free abelian, then $\Gamma$ is free in $\mathcal{V}$.*

**Proof.** After appealing to Lemma 3, it follows from Theorem A1 that we can find a free set $X = Y \cup Z$ of generators of $G$ such that $N = gp_G(Y)$. Hence $\Gamma$ is isomorphic to the subgroup of $G$ generated by $Z$. Consequently $\Gamma$ is free in $\mathcal{V}$. □

One of the consequences of Corollary 4 is Theorem A2, formulated below. Stallings' theorem will be deduced from it.

**Theorem A2.** *Let $\mathcal{V}$ be a variety of groups and let $G$ be a free group in $\mathcal{V}$. Let $N$ be a normal subgroup of $G$ satisfying the $\mathcal{H}$-condition and let $\Gamma = G/N$. Suppose that $G$ is residually nilpotent and that the abelianizations of $G$ and $\Gamma$ are free abelian. If the subset $Z$ of $G$ freely generates $G$ modulo $N[G,G]$, then the subgroup $E$ of $\Gamma$ generated by $\{zN \mid z \in Z\}$ is free in $\mathcal{V}$, freely generated by $\{zN \mid z \in Z\}$.*

The proof of Theorem A2 depends on the following lemma which is an analog of a theorem of Magnus [9] and which is proved by simply mimicking Magnus' proof:

**Lemma 5.** *Let $\mathcal{V}$ be a variety of groups in which the free groups are residually nilpotent. Suppose that $\Gamma$ is a group in $\mathcal{V}$ and that $Z$ is a subset of $\Gamma$ which freely generates $\Gamma$ modulo $[\Gamma,\Gamma]$. If $\Gamma/\gamma_n(\Gamma)$ is a free group in the variety of all nilpotent groups in $\mathcal{V}$ of class at most $n-1$, then the subgroup $E$ of $\Gamma$ generated by $Z$ is free in $\mathcal{V}$, freely generated by $Z$.*

**Proof.** Now

$$E/E \cap \gamma_n(\Gamma) \cong E\gamma_n(\Gamma)/\gamma_n(\Gamma) = \Gamma/\gamma_n(\Gamma).$$

Observe that $E/E \cap \gamma_n(\Gamma)$ is nilpotent of class at most $n-1$ which implies that $\gamma_n(E) \leq E \cap \gamma_n(\Gamma)$. On the other hand, $E \cap \gamma_n(\Gamma) \geq \gamma_n(E)$ and therefore $E \cap \gamma_n(\Gamma) = \gamma_n(E)$. It follows that $E/\gamma_n(E)$ is free in the variety of all nilpotent groups in $\mathcal{V}$ of class at most $n-1$ freely generated by $\{z\gamma_n(E) \mid z \in Z\}$. Let $F$ be a free group in $\mathcal{V}$, freely generated by a set $Z' = \{z' \mid z \in Z\}$ in a one-to-one correspondence $z' \mapsto z$ with $Z$. The map $\mu : z' \mapsto z$ induces then a homomorphism $\mu$ from $F$ onto $E$ and therefore a homomorphism $\mu_n$ of $F/\gamma_n(F)$ onto $E/\gamma_n(E)$. Since $E/\gamma_n(E)$ is freely generated by $\{z\gamma_n(E) \mid z \in Z\}$, $\mu_n$ is an isomorphism. Now if $a \in F$ and $a \neq 1$, then there exists an integer $n$ such that $a \notin \gamma_n(F)$. So $a\gamma_n(F) \neq 1$ which means its image under $\mu_n$ is also not equal to 1. It follows that $\mu$ is an isomorphism as required. □

Theorem A2 follows immediately from Lemma 5 since by Corollary 4 $\Gamma/\gamma_n(\Gamma)$ is free in the variety of nilpotent groups of class at most $n-1$ in $\mathcal{V}$.

We come next to the following generalization of Stallings' theorem.

**Corollary 6.** *Let $\mathcal{V}$ be a variety of groups and let $G$ be a free group in $\mathcal{V}$. Let $N$ be a normal subgroup of $G$ satifying the $\mathcal{H}$-condition and let $\Gamma = G/N$. Suppose that $G$ is residually nilpotent, that $G/\gamma_n(G)$ is torsion-free and that $\Gamma_{ab}$ is free abelian. If the subset $Z_1$ of $G$ freely generates modulo $N[G,G]$ a free abelian group, then the subgroup $E_1$ of $\Gamma$ generated by $\{zN \mid z \in Z_1\}$ is free in $\mathcal{V}$, freely generated by $\{zN \mid z \in Z_1\}$.*

Let $Z$ be a subset of $\Gamma$ which freely generates $\Gamma$ modulo its derived group. By Theorem A2 the subgroup $E$ of $\Gamma$ generated by $Z$ freely generates a free group in $\mathcal{V}$. Since the set $\{zN \mid z \in Z_1\}$ freely generates a free abelian group modulo the derived group of $\Gamma$, it freely generates a free abelian group modulo the derived group of $E$. Now by a theorem of Baumslag [3] any set of elements of $E$ which freely generates modulo its derived group a free abelian group, freely generates a free group in $\mathcal{V}$. So $E_1$ is free in $\mathcal{V}$, freely generated by $\{zN \mid z \in Z_1\}$.

If we now observe that absolutely free groups are residually nilpotent and that the quotients by the terms of the lower central series of a free group are torsion-free [10], then Stallings' theorem becomes a special case of Corollary 6.

Corollary 6 was first proved by Stammbach [13]. As noted earlier, his book [13] contains many results related to the work described here. It is worth pointing out, however, that our approach is quite different from anything dealt with in [13].

## 4. $\mathbb{Q}$-Groups

The basic idea in our approach to Stallings' theorem is best understood by working not in the category of all groups, but in the category of so-called $\mathbb{Q}$-groups. As noted in the introduction, with a little work many of the results about $\mathbb{Q}$-groups can be deduced from the corresponding results for ordinary groups and *vice-versa*. We will discuss this aspect of our work in Sec. 6.

### 4.1. $\mathbb{Q}$-*Groups*

**Definition 7.** A group $H$ is termed a $\mathbb{Q}$-group if every element of $H$ has a unique $n$th root, for every positive integer $n$.

Each such $\mathbb{Q}$-group $H$ can be viewed as a group which admits a set $\rho_n$ of unary operators, one for each positive integer $n$, satisfying the laws:

$$(h\rho_n)^n = h = (h^n)\rho_n \quad (h \in H).$$

So if $q = \ell/m \in \mathbb{Q}$, where here $\ell$ and $m > 0$ are integers, then $\mathbb{Q}$ acts on $H$ as follows:

$$h^q = (h^\ell)\rho_m.$$

$\mathbb{Q}$-groups can be thought of as groups with operators or as universal algebras. As such they constitute a category where the usual notions of morphism, subobject and quotient object apply. The kernels of morphisms will be termed here $\mathbb{Q}$-ideals and subobjects $\mathbb{Q}$-subgroups. A class $\mathcal{V}$ of $\mathbb{Q}$-groups which is closed under $\mathbb{Q}$-subgroups, epimorphisms and cartesian products is called a variety (of $\mathbb{Q}$-groups). In every such variety $\mathcal{V}$ of $\mathbb{Q}$-groups, we have free objects which we term free $\mathcal{V}$-groups. We refer the reader to the book by Cohn [4] for further details, which we here take for granted. We emphasize the fact that if $I$ is a $\mathbb{Q}$-ideal of the $\mathbb{Q}$-group $H$ and if $q \in \mathbb{Q}$, then the action of $\mathbb{Q}$ on $H/I$ is given by

$$(hI)^q = h^q I \quad (h \in H).$$

We denote the smallest $\mathbb{Q}$-subgroup of the $\mathbb{Q}$-group $G$ containing $X$ by $gp_{\mathbb{Q}}(X)$ and the smallest $\mathbb{Q}$-ideal of $G$ containing $X$ by $id_{\mathbb{Q}}(X)$. We shall need to be able to differentiate between groups and $\mathbb{Q}$-groups and we do so by omitting all mention of $\mathbb{Q}$ when working inside the category of all groups.

### 4.2. *Nilpotent $\mathbb{Q}$-groups*

If the $\mathbb{Q}$-group $H$ is nilpotent, then it turns out that every term of the lower central series of $H$ is a $\mathbb{Q}$-ideal of $H$ and so, in particular, $[H, H]$ is an ideal of $H$ (Baumslag [1]). Therefore $H_{ab} = H/[H, H]$ is an abelian $\mathbb{Q}$-group and as such can be viewed as a vector space over $\mathbb{Q}$, allowing us to invoke the terminology of vector spaces. We then term any pre-image in $H$ of a basis for $H_{ab}$, a $\mathbb{Q}$-basis for $H$. It is not hard to prove that if $Y$ is a $\mathbb{Q}$-basis of the nilpotent $\mathbb{Q}$-group $H$, then $Y$ $\mathbb{Q}$-generates $H$, i.e., $gp_{\mathbb{Q}}(Y) = H$.

The following lemma will be useful in the sequel.

**Lemma 8.** *A normal $\mathbb{Q}$-subgroup $I$ of a nilpotent $\mathbb{Q}$-group $H$ is an ideal of $H$.*

**Proof.** Consider the factor group $H/I$. Notice that $H/I$ is torsion-free. To see this, suppose that $h^n \in I$ for some $h \in H$ and $n > 0$. Since $I$ is closed under extraction of roots, $h \in I$, as needed. Now in a torsion-free nilpotent group $n$th-roots are unique whenever they exist — this is due to Kantorovich (see Kurosh [7]). It follows that $H/I$ is a $\mathbb{Q}$-group, which completes the proof of the lemma. □

### 4.3. *Theorem B1 and some of its consequences*

We are now in a position to formulate the following analog of Theorem A1 for $\mathbb{Q}$-groups.

**Theorem B1.** *Let $H$ be a nilpotent $\mathbb{Q}$-group and let $I$ be a $\mathbb{Q}$-ideal of $H$ satisfying the $\mathcal{H}$-condition, i.e.,*

$$[H, H] \cap I = [H, I].$$

*Then there exists a basis $X = Y \cup Z$ of $H$ such that $\{zI \,|\, z \in Z\}$ is a basis for $\Gamma = H/I$ and such that*

$$I = id_{\mathbb{Q}}(Y).$$

Before proceeding to the proof of Theorem B1 we note the following consequence of Theorem B1.

**Corollary 9.** *Let $H$ be a $\mathbb{Q}$-group, free in some variety $\mathcal{V}$ of nilpotent $\mathbb{Q}$-groups and let $I$ be an ideal of $H$ satisfying the $\mathcal{H}$-condition. Then $\Gamma = H/I$ is free in $\mathcal{V}$.*

**Proof of Corollary 9.** There exists, by Theorem B1, a $\mathbb{Q}$-basis $X = Y \cup Z$ for $H$ such that $I$ is the $\mathbb{Q}$-ideal of $H$ generated by $Y$. Hence $H/I \cong gp_{\mathbb{Q}}(Z)$ is also free in $\mathcal{V}$ on $\{zI \mid z \in Z\}$. □

Adopting the notation and hypothesis of Corollary 9, it follows that $H$ splits over $I$, with $gp_Q(Z)$ a complement of $I$ in $H$, substantiating the remarks made in Sec. 2.

We come now to the proof of Theorem B1.

**Proof.** The proof closely follows that of Theorem A1.

We view $H/[H,H]$ as a vector space over $\mathbb{Q}$ and $[H,H]I/[H,H]$ as a subspace of $H/[H,H]$. So we can choose a basis of $H/[H,H]$ which takes the form

$$\{z_o[H,H] \mid o \in O\} \cup \{y_m[H,H] \mid m \in M\},$$

where the $y_m \in I$ and constitute a basis for $I[H,H]/[H,H]$. Put

$$Z = \{z_o \mid o \in O\}, \quad Y = \{y_m \mid m \in M\} \text{ and } X = Y \cup Z.$$

Then $X$ $\mathbb{Q}$-generates $H$ modulo $[H,H]$ and hence $X$ $\mathbb{Q}$-generates $H$.

Let $I_1$ be the $\mathbb{Q}$-subgroup of $H$ generated by all of the conjugates of the elements of $Y$ by the elements of $H$. We claim that $I_1 = I$. Clearly $I_1 \leq I$. Our objective is to prove that the reverse inequality holds. The proof is by induction on the class $c$ of $H$. If $c = 1$, this is obvious. Assume then that $c > 1$. Now $\gamma_c(H)$ is an ideal of $H$; therefore following the proof of Theorem A1, we may assume inductively that $I \leq I_1\gamma_c(H)$. It follows that

$$[I,H] \leq [I_1\gamma_c(H), H] = [I_1, H].$$

Let $a \in I$. Then $a = bb'$, where $b \in I_1$ and $b' \in \gamma_c(H)$. Therefore $b' \in I \cap [H,H]$. Since $I \cap [H,H] = [I,H]$ and $[I,H] = [I_1,H]$, this implies that $b' \in [I_1,H]$ and therefore $b' \in I_1$. Consequently $a \in I_1$, i.e., $I \leq I_1$. This completes the proof of Theorem B1. □

## 5. Theorem B2

There is another rather less straightforward deduction that one can make from Theorem B1, namely the following analog of Stallings' theorem for $\mathbb{Q}$-groups:

**Theorem B2.** *Let $H$ be a $\mathbb{Q}$-group, which is residually torsion-free nilpotent. Suppose that $H$ is free in a variety $\mathcal{V}$ of $\mathbb{Q}$-groups and that $I$ is an ideal of $H$ satisfying the $\mathcal{H}$-condition. Furthermore, let $Z = \{z_o \mid o \in O\}$ be a subset of $H$ such that $Z' = \{z'_o = z_oI \mid o \in O\}$ is independent modulo the $\mathbb{Q}$-subgroup generated by the commutator subgroup of $\Gamma = H/I$. Then the $\mathbb{Q}$-group generated by $Z'$ is again free in $\mathcal{V}$. In particular if $\Gamma = gp_{\mathbb{Q}}(Z')$, then $\Gamma$ is free on $Z'$.*

Theorem B2 applies in particular to the variety of metabelian $\mathbb{Q}$-groups, since Ledlie [8] has proved that the free groups in this variety are residually torsion-free nilpotent.

### 5.1. *Proof of Theorem B2*

We denote by $\mathcal{E}$ the class of all groups in which every element has at least one $n$th root for every positive integer $n$. $\mathcal{E}$-groups are often referred to as divisible groups. In order to proceed, we need to provide some additional information about nilpotent $\mathcal{E}$-groups. To begin, we recall the following lemma of Kantorovich (see [7]).

**Lemma 10.** *Let $G$ be a nilpotent $\mathcal{E}$-group. Then the torsion subgroup $tor(G)$ of $G$ is central in $G$.*

We are now able to prove the following proposition, which is interesting in its own right.

**Proposition 11.** *Let $G$ be a nilpotent $\mathcal{E}$-group. Then $G$ splits as a direct product*

$$G = A \times B,$$

*where $B$ is contained in the torsion subgroup $tor(G)$ of $G$ and the torsion subgroup of $tor(A)$ of $A$ is contained in its derived group.*

**Proof.** Let $\eta$ be the canonical homomorphism of $G$ onto its abelianization $G_{ab}$. Since $G \in \mathcal{E}$, it follows also that $G_{ab} \in \mathcal{E}$. Let $K$ be the kernel of the restriction of $\eta$ to $tor(G)$. Now $K = tor(G) \cap [G, G]$. Since $G$ is a nilpotent $\mathcal{E}$-group, it follows, by induction and the usual commutator identities (see e.g. [1]) that $[G, G] \in \mathcal{E}$. Moreover, since $G \in \mathcal{E}$, it follows also not only that $tor(G) \in \mathcal{E}$, but that $tor(G)$ is isolated in $G$, i.e., closed under extraction of $n$th roots for every choice of the positive integer $n$. Hence $tor(G) \cap [G, G] \in \mathcal{E}$, i.e., $K \in \mathcal{E}$. Therefore since a divisible subgroup of an abelian group is a direct factor, $K$ is a direct factor of $tor(G)$:

$$tor(G) = K \times B.$$

It follows that $\eta$, restricted to $B$, is a monomorphism. Decompose $G_{ab}$ into a direct product

$$G_{ab} = C \times B\eta.$$

Let $\pi$ be the projection of $G_{ab}$ onto $B\eta$ and let $\nu$ be the inverse of $\eta$ restricted to $B$, i.e., the map which sends $b\eta$ ($b \in B$) to $b$. Finally let $A$ be the kernel of the homomorphism $\beta = \eta\pi\nu$. We claim that

$$G = A \times B.$$

Now $\beta$ is the identity on $B$. Hence $A \cap B = 1$. Moreover $B$ is central in $G$ and so $A$ and $B$ commute elementwise, which implies that they generate their direct product. Finally, if $g \in G$, then, appealing to a well-known argument,

$$g = (g(g\beta)^{-1})g\beta \in A \times B,$$

since $(g(g\beta)^{-1})\beta = 1$. This completes the proof of Proposition 11. □

We need some additional lemmas before we can proceed to the proof of Theorem B2.

**Lemma 12.** *Suppose that $G$ is a nilpotent $\mathbb{Q}$-group and suppose that $N$ is a normal subgroup of $G$. If $[N,G]$ is contained in the center of $G$, then $[N,G]$ is an ideal of $G$.*

**Proof.** Since $[N,G]$ is a normal subgroup of $G$, it suffices to prove that $[N,G]$ is isolated in $G$. Suppose then that $g \in G$, that $n$ is a positive integer and that $g^n \in [N,G]$. Then

$$g^n = [a_1, g_1] \cdots [a_k, g_k]$$

where $a_1, \ldots, a_k \in N$ and $g_1, \ldots, g_k \in G$. Now $G$ is an $\mathcal{E}$-group; so there exist for each of the $g_i$ elements $h_i \in G$ such that $g_i = h_i^n$. Each of the commutators $[a_i, h_i]$ is central in $G$; consequently

$$[a_i, g_i] = [a_i, h_i^n] = [a_i, h_i]^n$$

and therefore

$$g^n = [a_1, h_1]^n \cdots [a_k, h_k]^n = ([a_1, h_1] \cdots [a_k, h_k])^n.$$

Since extraction of roots in $G$ is unique, we find that

$$g = [a_1, h_1] \cdots [a_k, h_k] \in [N, G]$$

which completes the proof. □

**Corollary 13.** *Suppose that $G$ is a nilpotent $\mathbb{Q}$-group and that $M$ is a normal subgroup of $G$. Then $[M,G]$ is an ideal of $G$.*

**Proof.** Since $G$ is nilpotent, there exists an integer $\ell$ such that

$$\big[M, \underbrace{G, \ldots, G}_{\ell}\big] \neq 1, \quad \big[M, \underbrace{G, \ldots, G}_{\ell+1}\big] = 1.$$

By Lemma 12, $N = \big[M, \underbrace{G, \ldots, G}_{\ell}\big]$ is an ideal of $G$. So inductively we can assume that $[M,G]/N$ is an ideal of $G/N$. It follows that if $g \in [M,G]$ and if $n$ is a positive integer, then there exists an element $h \in [M,G]$ and an element $a \in N$ such that $g = h^n a$. Since $N$ is an ideal of $G$, there exists an element $b \in N$ such that $b^n = a$. Consequently $g = h^n a = h^n b^n = (hb)^n$. Since $hb \in [M,G]$, this completes the proof. □

We will find it useful to denote the isolator of a subgroup $H$ of a group $G$ by $\overline{H}$; so, by definition, $\overline{H}$ consists of those elements of $G$ with a non-trivial power in $H$. In the event that $G$ is nilpotent, $\overline{H}$ is again a subgroup of $G$ (cf. e.g. [7]). Notice that the $\mathbb{Q}$-subgroup generated by $[G,G]$ is $\overline{[G,G]}$.

Next we have

**Corollary 14.** *Suppose that $M$ is a normal subgroup of the nilpotent $\mathbb{Q}$-group $G$. Then*

$$\overline{[M,G]} = [M,G] = [\overline{M}, G].$$

**Proof.** We have already proved that $[M, G]$ is an ideal of $G$. Since it contains $[M, G]$ it also contains $\overline{[M, G]}$. On the other hand $\overline{[M, G]}$ consists of all elements of $G$ which have a non-trivial power in $[M, G]$. Therefore $\overline{[M, G]} \geq [M, G]$. So we have proved that $\overline{[M, G]} = [M, G]$.

It remains to prove that $[M, G] = [\overline{M}, G]$. Now $G/[M, G]$ is a $\mathbb{Q}$-group. Every element of $\overline{M}$ has a non-trivial power in $M$. Now Kantorovich has proved that the center of a group in which extraction of roots is unique is isolated [7]. This implies that modulo $[M, G]$, $G$ is centralized by $\overline{M}$. In other words, $\overline{M}$ is contained in the center of $G$ modulo $[M, G]$ which means that $[\overline{M}, G] \leq [M, G]$. The reverse inequality follows from the fact that $M \leq \overline{M}$. This completes the proof of Corollary 14. □

We are now in a position to prove the key step in the proof of Theorem B2.

**Lemma 15.** *Suppose that $R = S \times T$, where $S$ is a nilpotent $\mathbb{Q}$-group and $T$ is a torsion, abelian group. Furthermore, suppose that $U$ is a normal subgroup of $R$ satisfying the $\mathcal{H}$-condition. If $\pi$ is the projection of $R$ onto $S$, if $P = U\pi$ and if $\overline{P}$ is the isolator of $P$ in $S$, then*

$$[S, S] \cap \overline{P} = [S, \overline{P}],$$

*i.e., $\overline{P}$, viewed as a subgroup of $S$, satisfies the $\mathcal{H}$-condition.*

**Proof.** Since $U$ satisfies the $\mathcal{H}$-condition,

$$[R, R] \cap U = [R, U]. \tag{1}$$

It then follows from (1) that

$$[S, S] \cap U = [S, P]. \tag{2}$$

We claim that

$$[S, S] \cap \overline{P} = [S, \overline{P}], \tag{3}$$

We prove first that

$$[S, S] \cap \overline{P} \leq [S, \overline{P}]. \tag{4}$$

Suppose that $a \in [S, S] \cap \overline{P}$. Since $a \in \overline{P}$ there exists a positive integer $n$ such that $a^n \in P$. So $a^n \in [S, S] \cap P$. Furthermore, because $a^n \in P$, there exists an element $t \in T$ such that $a^n t \in U$. $T$ is a torsion group. Consequently there exists a positive integer $m$ such that $t^m = 1$. It follows that $a^{mn} \in U$. Hence $a^{mn} \in [S, S] \cap U$. Consequently by (2) $a^{mn} \in [S, P]$. Since $P \leq \overline{P}$, $a^{mn} \in [S, \overline{P}]$. Now by Lemma 12, $[S, \overline{P}]$ is an ideal of $S$. This implies then that $a \in [S, \overline{P}]$. This completes the proof of (4). □

Now, using Corollary 14 to begin with, we find that

$$[S, \overline{P}] = [S, P] = [S, U] = [R, U] = [R, R] \cap U \leq [S, S] \cap P \leq [S, S] \cap \overline{P}.$$

This completes the proof of Lemma 15.

There are two more steps needed in order to complete the proof of Theorem B2. The first is the following lemma.

**Lemma 16.** *Let $H$ be a free group in a variety $\mathcal{V}$ of $\mathbb{Q}$-groups. Suppose that $H$ is residually torsion-free nilpotent and that $I$ is a $\mathbb{Q}$-ideal of $H$ satisfying the $\mathcal{H}$-condition. Furthermore suppose that $Z = \{z_o \mid o \in O\}$ is a subset of $H$ and suppose that $\{z_o I \mid o \in O\}$ is linearly independent modulo the isolator of the derived subgroup of $\Gamma = H/I$. Then there exists for each $n > 1$ an ideal $J_n$ of $H$ such that*

(1) *$H/J_n$ is free in the variety $\mathcal{V}_n$ of all nilpotent groups of class at most $n-1$ in $\mathcal{V}$;*
(2) *$J_n \leq \overline{[H,H]I}$ and so the elements $z_o J_n$ $o \in O\}$ are independent modulo the derived group of $H/J_n$.*

**Proof.** Now by Proposition 11, $H/\gamma_n(H)$ splits as a direct product

$$H/\gamma_n(H) = A_n/\gamma_n(H) \times B_n/\gamma_n(H),$$

where $B_n/\gamma_n(H)$ is a torsion group and the torsion subgroup $tor(A_n/\gamma_n(H)) = C_n/\gamma_n(H)$ of $A_n/\gamma_n(H)$ is contained in its derived group. It follows in particular that $\overline{\gamma}_n(H) = C_n \times B_n$ and also that

$$H/\overline{\gamma}_n(H) = A_n/C_n \times B_nC_n/C_n \cong A_n/C_n.$$

By Lemma 2, $I\gamma_n(H)/\gamma_n(H)$ satisfies the $\mathcal{H}$-condition and again by Lemma 2, $IC_n/C_n$, now viewed as a subgroup of $H/C_n$, satisfies the $\mathcal{H}$-condition. Observe that $H/C_n = A_n/C_n \times B_nC_n/C_n$ and that $A_n/C_n$ is a nilpotent $\mathbb{Q}$-group, indeed isomorphic to $H/\overline{\gamma}_n(H)$, which is free in $\mathcal{V}_n$. Now the elements $z_o(C_nB_n) = z_o\overline{\gamma}_n(H)$ are linearly independent modulo the derived group of $H/\overline{\gamma}_n(H)$. Let now $\pi_n$ be the projection of $H/C_n$ onto $A_n/C_n$. Then by Lemma 15, $\overline{(IC_n/C_n)\pi_n} = K_n/C_n$ is an ideal of the $\mathbb{Q}$-group $A_n/C_n$ satisfying the $\mathcal{H}$-condition. It follows also that $K_nB_n/C_nB_n$ is a subgroup of $H/C_nB_n$ satisfying the $\mathcal{H}$-condition. We have already noted that $C_nB_n = \overline{\gamma}_n(H)$. So, putting $K_nB_n = J_n$ we find that $J_n/\overline{\gamma}_n(H)$ is a $\mathbb{Q}$-subgroup of $H/\overline{\gamma}_n(H)$ satisfying the $\mathcal{H}$-condition. So by Corollary 9, $(H/\overline{\gamma}_n(H))/(J_n/\overline{\gamma}_n(H)) \cong H/J_n$ is free in $\mathcal{V}_n$. Moreover $J_n \leq \overline{[H,H]I}$ and therefore the elements $z_oJ_n$ ( $o \in O$) are independent modulo the derived group of $H/J_n$. Consequently they freely generate, modulo $J_n$ a free group in $\mathcal{V}_n$. This completes the proof of Lemma 16. □

In order to complete the proof of Theorem B2 we need a corollary to the following analog of Lemma 5 for $\mathbb{Q}$-groups.

**Lemma 17.** *Let $\mathcal{V}$ be a variety of $\mathbb{Q}$-groups in which the free groups are residually torsion-free nilpotent. Suppose that $\Gamma$ is a group in $\mathcal{V}$ and that $X$ is a subset of $\Gamma$ which freely generates $\Gamma$ modulo $\overline{[\Gamma,\Gamma]}$. If there exists for each $n$ a quotient of $\Gamma$ which is free in the variety of all nilpotent $\mathbb{Q}$-groups, freely generated by the image of $X$, then the $\mathbb{Q}$-group $H$ generated by $X$ is free in $\mathcal{V}$, freely generated by $X$.*

**Proof.** Let $F$ be a $\mathbb{Q}$-group, free in $\mathcal{V}$ on a set $X' = \{y' \mid x \in X\}$, in a one-to-one correspondence with $X$. Let $\phi$ be the homomorphism of $F$ onto $H$ defined by mapping $x' \in X'$ to $x \in X$. This homomorphism induces a homomorphism from $F/\bar{\gamma}_n(F)$ onto $H/\bar{\gamma}_n(H)$. Now according to the hypothesis, $H/\bar{\gamma}_n(H)$ maps via a homomorphism, say $\delta$, onto a free nilpotent group $E$ in the variety of all nilpotent $\mathbb{Q}$-groups of class at most $n-1$ in $\mathcal{V}$, freely generated by $X\delta$. Let $\phi$ be the homomorphism of $E$ onto $H/\bar{\gamma}_n(H)$ defined by mapping $x\delta$ back to $x$. It follows that $\delta\phi$ is the identity on $E$ and therefore that $H/\bar{\gamma}_n(H)$ is free nilpotent of class $n-1$ in the variety of all nilpotent groups of class at most $n-1$ in $\mathcal{V}$. □

This brings us to the last step in the proof of Theorem B2.

**Corollary 18.** *Suppose that $\mathcal{V}$ is a variety of $Q$-groups in which the free groups are residually torsion-free nilpotent. Let $G$ be a free group in $\mathcal{V}$. If $Y$ is any subset of $G$ which is independent modulo the $\mathbb{Q}$-subgroup of $G$ generated by $[G,G]$, then $gp_{\mathbb{Q}}(Y)$ is again free in $\mathcal{V}$, freely generated by $Y$.*

**Proof.** We can extend $Y$ to a set $X$ such that $X$ is a basis for $G$ modulo $J = \overline{[G,G]}$. Let $H$ be the $\mathbb{Q}$-group generated by $X$. Then $HJ = G$ and hence $H\bar{\gamma}_n(G) = G$. It follows that

$$H/\bar{\gamma}_n(G) \cap H \cong G/\bar{\gamma}_n(G).$$

Hence, by Lemma 17, $H$ is free in $\mathcal{V}$ and freely generated by $X$. Since $Y$ is a subset of $X$ it follows that $gp_{\mathbb{Q}}(Y)$ is also free in $\mathcal{V}$ and freely generated by $Y$. □

Theorem B2 follows immediately. Indeed, if we put Lemma 16, Lemma 17 and Corollary 18 together, we find that any subset of $\Gamma$ which is independent modulo the $\mathbb{Q}$-subgroup generated by the commutator subgroup of $\Gamma = H/I$, freely generates a free group in $\mathcal{V}$.

## 6. Malčev Completions and the $\mathcal{H}$-Condition

Every torsion-free nilpotent group $G$ can be embedded in a nilpotent $\mathbb{Q}$-group $H$. If $gp_{\mathbb{Q}}(G) = H$, then $H$ turns out to be unique. This follows from the fact that if $\phi$ is a homomorphism of $G$ into a nilpotent $\mathbb{Q}$-group $K$, then $\phi$ can be continued to a homomorphism from $H$ into $K$. It follows that if $G$ is a subgroup of the $\mathbb{Q}$-group $K$ and if $gp_{\mathbb{Q}}(G) = K$, then there is an isomorphism from $H$ to $K$ which extends the identity map from $G$, qua subgroup of $H$, to $G$, qua subgroup of $K$. Such a group $H$ is termed the Malčev completion of $G$, which we denote also by $m(G)$. One of the properties of $H$ is that given an element $h \in H$, there exists a positive integer $j$ such that $h^j \in G$.

**Lemma 19.** *Let $G$ be a torsion-free nilpotent group and let $N$ be a normal subgroup of $G$. Then $m(N)$ is an ideal of $m(G)$.*

**Proof.** It suffices by Lemma 8 to prove that $m(N)$ is a normal subgroup of $m(G)$. Let $H = m(G)$. Notice that the class $c$ of $H$ coincides with the class of $G$. The proof then is by induction $c$. If $c = 1$ there is nothing to prove. Suppose then that $c > 1$ and that modulo the center $C$ of $H$, $m(N)$ is normal in $H$. Now let $a \in m(N)$ and let $b \in H$. Then inductively $b^{-1}ab \in m(N)C$. So $b^{-1}ab = a'y$, where $a' \in m(N)$ and $y \in C$. There exists a positive integer $n$ such that $a^n \in G$, $a'^n \in N$ and $b^n \in G$ which implies that

$$b^{-n}a^nb^n = a'^ny^n \in N.$$

This yields $y^n \in N$ and so $y \in m(N)$. Hence $b^{-1}ab = a'y$ where $a'y \in m(N)$ and this completes the proof of Lemma 17. □

Lemma 17 allows us to prove the following:

**Lemma 20.** *Let $G$ be a torsion-free nilpotent group and let $N$ be a normal subgroup of $G$ which satisfies the $\mathcal{H}$-condition. Then $m(N)$ is an ideal of $m(G)$ which satisfies the $\mathcal{H}$-condition.*

**Proof.** By Lemma 17, $m([G, G])$ is an ideal of $m(G)$. If $a, b \in m(G)$ then some non-trivial power of both $a$ and $b$ lies in $G$ and hence modulo $m([G, G])$ the elements $a$ and $b$ commute. So $m(G)/m([G, G])$ is abelian. So $[m(G), m[G]] \leq m([G, G])$. Now $[m(g), m(g)]$ is an ideal of $m(G)$ which clearly contains $[G, G]$. So $[m(G), m[G]] \geq m([G, G])$. So we have proved that $[m(G), m[G]] = m([G, G])$. It follows that $[m(G), m(G)] \cap m(N)$ consists of those elements of $m(G)$ with a non-trivial power in $[G, G] \cap N = [G, N]$. Therefore $[m(G), m(G)] \cap m(N) \leq m([G, N])$. Similarly we find that $m([G, N])$ consists of those elements of $m(G)$ with a non-trivial power in $[G, N] = [G, G] \cap N$. Consequently $m([G, N]) \leq m([G, G] \cap m(N))$, which completes the proof of Lemma 12. □

Lemma 17 enables us to deduce Theorem A1 from Theorem B1. Many of the subgroup theorems can then also be deduced one from the other by appealing to Theorem C, which will be formulated and proved in Sec. 7, below.

## 7. Connecting Varieties

In this section we briefly discuss, as we indicated earlier, some of the connections between torsion-free nilpotent groups and nilpotent $\mathbb{Q}$-groups. We shall have need here for two varieties, the variety $\mathcal{V}$ of $\mathbb{Q}$-groups generated by the Malčev completion $m(G)$ of a torsion-free nilpotent group $G$ and the variety $\mathcal{U}$ of groups generated by $G$. The following theorem then holds.

**Theorem C.**

(1) *Let $G_1$ be a torsion-free nilpotent group and let $G$ be a subgroup of $G_1$. If every element of $G_1$ has a positive power in $G$, then $G$ and $G_1$ generate the same variety.*

(2) *If $G$ is a free group in a variety of nilpotent groups and if $T$ is the torsion subgroup of $G$, then $m(G/T)$ is free in the variety $\mathcal{V}$ of $\mathbb{Q}$-groups generated by $m(G/T)$.*

(3) *Let $H$ be a nilpotent $\mathbb{Q}$-group, free in a variety $\mathcal{V}$ of nilpotent $\mathbb{Q}$-groups, freely $\mathbb{Q}$-generated by $X$ and let $\mathcal{U}$ be the variety of groups generated by $H$. Then $G = gp(X)$ is free in $\mathcal{U}$, freely generated by $X$.*

(4) *If $G$ is a torsion-free nilpotent group, free in a variety $\mathcal{U}$ and if $\mathcal{V}$ is the variety of $\mathbb{Q}$-groups generated by $m(G)$, then $m(G)$ is free in $\mathcal{V}$.*

**Proof.** (1) Since a variety is completely determined by the laws that hold in all the groups in the variety, it suffices to restrict attention to the case where $G_1$ is finitely generated. It follows then that $G$ is of finite index, say $j$, in $G_1$ (see [7]). Let $p$ be a prime which does not divide $j$. Then, according to a theorem of Gruenberg [5], $G_1$ is residually a finite $p$-group and hence is a subdirect product of quotients of $G_1$ of $p$-power order. Since $G$ is of index $j$ in $G_1$, the image of $G$ in each such quotient $K$ coincides with $K$. It follows that $G_1$ is a subdirect product of quotients of $G$ and hence is contained in the variety generated by $G$. This completes the proof of (1).

(2) Let $\mathcal{U}$ be the variety of groups generated by $G' = G/T$. Since $G$ is free in some variety of groups, we can express $G$ as a quotient of an absolutely free group $F$, freely generated by a set $X$, by a fully invariant subgroup $I$ of $F$:

$$G \cong F/I.$$

This isomorphism induces an isomorphism between $T$ and a subgroup $J/I$ of $F/I$:

$$T \cong J/I.$$

Notice that $J$ is a fully invariant subgroup of $F$ and since $G' \cong F/J$, $G'$ is free in $\mathcal{U}$. The laws defining $\mathcal{U}$ are simply the $X$-words which comprise $J$. It follows that $G'$ is free in $\mathcal{U}$ on $X' = \{xJ \mid x \in X\}$. Hence every mapping of $X'$ into a group $H$ in $\mathcal{U}$ can be extended to a homomorphism of $G'$ into $H$. Now by (1) every group in $\mathcal{V}$ is contained in $\mathcal{U}$. Hence every mapping of $X'$ into a $\mathbb{Q}$-group $H$ in $\mathcal{V}$ can be extended to a homomorphism. Now $m(G')$ has the property that every homomorphism of $G'$ into a nilpotent $\mathbb{Q}$-group $H$ can be extended to a homomorphism of $m(G')$ into $H$ (A. I. Malčev [11]). Hence, noting that $gp_{\mathbb{Q}}(X') = m(G')$, we find that every mapping of $X'$ into a group $H$ in $\mathcal{V}$ can be extended first to a homomorphism from $G'$ into $H$ and then to a homomorphism of $m(G')$ into $H$. This means that $m(G')$ is free on $X'$ in $\mathcal{V}$.

(3) Let $F$ be a free group in $\mathcal{U}$, freely generated by a set $X' = \{x' \mid x \in X\}$, in a one-to-one correspondence with $X$. The mapping from $X'$ to $X$ defined by $x' \mapsto x(x \in X)$ can be extended to a homomorphism $\mu$ of $F$ into $H$. Notice that $\mu$ maps $F$ onto $G$. On the other hand, suppose that $w(x_1, \ldots, x_n)$ is a product of the elements $x_i \in X$ and their inverses. If $w(x_1, \ldots, x_n) = 1$ in $G$, then $w = 1$ is a law in $\mathcal{V}$ which does not involve the action of $\mathbb{Q}$ and so it is also a law in $\mathcal{U}$. Hence $w(x'_1, \ldots, x'_n) = 1$ in $F$. Thus the mapping from $X$ to $X'$ sending each $x \in X$ to

$x' \in X'$ defines a homomorphism $\eta$ of $G$ onto $F$. But $\mu$ and $\eta$ are inverses and so $G$ is free on $X$ as claimed.

(4) Suppose that $G$ is freely generated by $X$. Then $X$ $\mathbb{Q}$-generates $m(G)$. Now let $E$ be a $\mathbb{Q}$-group in $\mathcal{V}$ and let $\alpha$ be a map from $X$ into $E$. By (1), every group in $\mathcal{V}$ is contained in $\mathcal{U}$. Hence $\alpha$ can be continued to a homomorphism $\beta$ from $G$ into $E$. But then $\beta$ itself can be continued to a homomorphism from $m(G)$ into $E$, since $m(G)$ is the Malčev completion of $G$. □

## 8. Lie Algebras

There are analogs of all of the results described above for Lie and associative algebras over fields and also over $\mathbb{Z}$. We shall not go into any of the details here, leaving them (and the appropriate transcriptions) to the interested reader. We content ourselves with the fomulation of one such analog here, adopting the usual bracket notation for Lie algebras as used for example in Jacobson's book [6].

**Theorem D.** *Let $F$ be a free lie algebra over $k$, a field of characteristic 0 and let $L = F/I$ be a quotient of $F$ by an ideal $I$. If*

$$[F, F] \cap I = [F, I],$$

*then any subset $Z$ of $L$ which is independent modulo the derived Lie algebra $[L, L]$ of $L$, freely generates a subalgebra of $L$ which is free on $Z$.*

It is probable that Theorem D is not new, but I have been unable to find it in the literature. The main point in this regard is that the approach taken here can be mimicked in the category of Lie algebras as well as in the category of associative algebras. In both these categories there are analogs of most of our theorems and which also have many consequences. In addition all of Stammbach's results can be re-proved in these slightly different settings.

## Acknowledgment

The author was supported in part by NSF Grant #9626231.

## References

[1] G. Baumslag, Some aspects of groups with unique roots, *Acta Math.* **104** (1960) 217–303.
[2] G. Baumslag, Groups with the same lower central series as a relatively free group. I. The groups, *Trans. Amer. Math. Soc.* **142** (1969) 507–538.
[3] G. Baumslag, Some subgroup theorems, *Trans. Amer. Math. Soc.* **108** (1963) 516–525.
[4] P. M. Cohn, *Universal Algebra* (Harper & Row, New York, 1969).
[5] K. W. Gruenberg, Residual properties of infinite soluble groups, *Proc. London Math. Soc.* **7** (1957) 29–62.
[6] N. Jacobson, *Lie Algebras* (Interscience, New York, 1969).
[7] A. G. Kurosh, *The Theory of Groups*, Vols. 1 and 2 (reprinted by Chelsea).

[8] J. Ledlie, Free metabelian $D$-groups, *Trans. Amer. Math. Soc.* **17** (1978) 427–436.
[9] W. Magnus, Ueber freie faktorgruppen und freie untergruppen gegebener gruppen, *Monatsh. Math. Phys.* **47** (1939) 307–313.
[10] W. Magnus, Ueber beziehungen zwischen hoehren kommutatoren, *J. Reine Angew. Math.* **177** (1937) 105–117.
[11] A. I. Malčev, Nilpotent torsion free groups, *Izv. Akad. Nauk. SSSR Ser. Mat.* **13** (1949) 201–212.
[12] J. R. Stallings, Homology and central series of groups, *J. Algebra* **2** (1965) 170–181.
[13] U. Stammbach, *Homology in Group Theory*, Lecture Notes in Mathematics, Vol. 359 (Springer-Verlag, Berlin, Heidelberg, New York, 1973).