Unsolvable problems about small cancellation and word hyperbolic groups

G. Baumslag, C. F. Miller III and H. Short

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Abstract

We apply a construction of Rips to show that a number of algorithmic problems concerning certain small cancellation groups and, in particular, word hyperbolic groups, are recursively unsolvable. Given any integer $k > 2$, there is no algorithm to determine whether or not any small cancellation group can be generated by either two elements or more than k elements. There is a small cancellation group E such that there is no algorithm to determine whether or not any finitely generated subgroup of E is all of E, or is finitely presented, or has a finitely generated second integral homology group.

Introduction

In [6] Gromov introduced the notion of a *word hyperbolic group* and gave a number of equivalent characterizations, among them that word hyperbolic groups are the finitely presented groups which satisfy a so-called linear isoperimetric inequality (see [6] for definitions and details). The class of word hyperbolic groups includes all finite groups, free groups, certain small cancellation groups ([5] for a detailed discussion of this subject) and fundamental groups of closed negatively curved Riemanian manifolds (see [6]).

From an algorithmic point of view, word hyperbolic groups have some nice properties. In particular, Gromov [6] has shown that word hyperbolic groups have solvable word problem and solvable conjugacy problem. Indeed, essentially the same algorithms described by Dehn to solve the word and conjugacy problems for surface groups can be used to solve the corresponding problems for word hyperbolic groups.

Notice that since word hyperbolic groups have solvable word problem, the property *being a word hyperbolic group* is a Markov property (see [10] or [8]). Hence there is no algorithm to decide whether or not an arbitrary finite presentation defines a word hyperbolic group. However, given a collection of groups known in advance to be word hyperbolic, one might hope to find algorithms to answer various questions about them. For instance one might ask whether there is an algorithm which determines whether or not any of the groups in the class is finite. In general, i.e. for finitely presented groups as a whole, there is no such algorithm (see e.g. [10]). But there is one for hyperbolic groups. We will sketch a proof below. Perhaps the most interesting and as yet unresolved problem about hyperbolic groups is the isomorphism problem: given a recursive family of finite presentations of groups all of which are known to be hyperbolic, is there an algorithm which decides whether or not any pair of these presentations define isomorphic groups? Our first result, cited in the abstract, suggests that this may well be a difficult problem. We will also prove here that a number of other interesting decision problems for many small cancellation groups, including word hyperbolic groups, are recursively unsolvable. Most of these have been cited in the abstract. A key ingredient in our proofs is a clever construction introduced by Rips [12] which shows that every finitely presented group is a quotient of a small cancellation group by a finitely generated normal subgroup. Indeed, Rips already observed that the generalized word problem for small cancellation groups is unsolvable.

The finiteness problem

The class of hyperbolic groups is contained in a slightly larger class of groups, termed automatic groups, which were introduced in [4]. We will take the definition of these groups, as well as most of their properties, for granted here. Now given a recursive class of finite presentations of automatic groups, there is an uniform algorithm which yields an automatic structure for each of the groups in the given class [4]. Moreover there is a uniform algorithm which decides whether or not any automatic group is finite (see e.g., [2]). So, in slightly less precise terms, this means that there is an algorithm which determines whether or not any automatic, and hence also any hyperbolic group, is finite.

The Rips construction

We review the essential properties of the Rips construction. Suppose that the finitely presented group G is given by the finite presentation $P =< x_1, \ldots, x_s \mid r_1 = 1, \ldots, r_t = 1 >$. Let $\rho = \rho_P$ be a positive integer greater than the maximum of the lengths of the r_i. Introduce two new symbols a and b and define the words

$$z(a, b, k) = (ab)^{80\rho k+1}(ab^2)^{80\rho k+2} \cdots (ab)^{80\rho k+79}(ab^2)^{80\rho k+80}.$$

(There are many alternative ways to choose an appropriate collection of words for this construction.) Let E_P be the group with the following finite presentation: the generators of E_P are $a, b, x_1, \ldots, x_s$; the relations of E_P are

$$\begin{aligned}
x_i^{-1} a x_i &= z(a,b,i), i = 1, \ldots, s; \\
x_i a x_i^{-1} &= z(a,b,s+i), i = 1, \ldots, s; \\
x_i^{-1} b x_i &= z(a,b,2s+i), i = 1, \ldots, s; \\
x_i b x_i^{-1} &= z(a,b,3s+i), i = 1, \ldots, s; \\
r_j &= z(a,b,4s+j), j = 1, \ldots, t
\end{aligned}$$

Denote by K_P the subgroup of E_P generated by a, b. Then the folowing properties are easily verified:

1. K_P is a normal subgroup of E_P and the quotient E_P/K_P is isomorphic to G, that is, we have an exact sequence $1 \to K_P \to E_P \xrightarrow{\gamma} G \to 1$.

2. The given presentation for E_P satisfies the metric small cancellation condition $C'(\frac{1}{10})$ (see [8]). Hence E_P is a word hyperbolic group.

3. E_P has solvable word problem and solvable conjugacy problem. Further, E_P is an aspherical group and has cohomological dimension 2 (see [8] for these assertions).

Because it typifies the application of this construction, we repeat the following unsolvability result already noted by Rips [12]:

Corollary 1 (Rips) *There is a small cancellation group E, with a finite presentation which satisfies the small cancellation condition $C'(\frac{1}{10})$, and which contains a finitely generated normal subgroup K such that the problem of deciding of an arbitrary word w of E whether or not w represents an element of K is recursively unsolvable. That is, the generalized word problem for K in E is unsolvable.*

Proof: Suppose that G, in the construction above, is a group with unsolvable word problem. Put $E = E_P$ and $K = K_P$. Let w be an arbitrary word in the x_i. Then w as a word in E represents an element of K if and only if $w =_G 1$. Since G has unsolvable word problem the result follows.

Some unsolvable problems

We now apply the Rips construction in various ways to obtain a number of unsolvability results. We will need to appeal to the unsolvability of various problems for finitely presented groups in general. These topics are surveyed in [10]. Recall that the *rank* of a group is the minimum number of generators in any presentation.

Theorem 2 *Given any integer $k > 2$, there is a recursive class of groups given by finite presentations which satisfy the $C'(\frac{1}{10})$ small cancellation condition, such that there is no algorithm which determines whether any group in the class can be generated by either two elements or more than k elements. Hence, there is no algorithm to determine the rank of a finitely presented small cancellation group and, in particular, there is no algorithm to determine the rank of any hyperbolic group.*

Proof: Fix a positive integer $k \geq 3$. If P is a finite presentation of a group G, denote by P^{*k} the presentation of the k-fold ordinary free product $G^{*k} =_{def} \underbrace{G * G * \cdots * G}_{k}$ obtained by joining together k copies of P on disjoint alphabets. By the Grushko-Neumann Theorem, if G is not the trivial group, the rank of G^{*k} is at least $k \geq 3$.

Next apply the Rips construction to obtain presentations for the groups $E_{P^{*k}}$ as above. Recall that G^{*k} is a quotient group of $E_{P^{*k}}$. So if G is not the trivial group, then the rank of $E_{P^{*k}}$ is at least $k \geq 3$. But if G is the trivial group, then P^{*k} is also a presentation of the trivial group and $E_{P^{*k}}$ is generated by a and b and so has rank 2. Since there is no algorithm to decide of an arbitrary finite presentation P whether or not P presents the trivial group [11], it follows that there is no algorithm to determine whether or not $E_{P^{*k}}$ can be generated by fewer than 3 of its elements. This completes the proof.

In preparation for our next unsolvability result, we need the following:

Lemma 3 *Let P be a finite presentation of a group G. Suppose that L is a finitely generated subgroup of G and denote its preimage $\gamma^{-1}(L)$ in E_P by $\widetilde{L}$.*

1. (Rips) *If $\widetilde{L}$ is finitely presented, then L is finitely presented.*
2. *If G has cohomological dimension 2 and if $H_2(\widetilde{L}, \mathbf{Z})$ is finitely generated, then $H_2(L, \mathbf{Z})$ is finitely generated.*

Proof: Observe that $\widetilde{L}$ is generated by a finite set of elements consisting of a and b together with one pre-image of each generator of L. Suppose now that $\widetilde{L}$ could be finitely presented. We can assume that a and b are among the generators of this presentation. Then L has finite presentation obtained from that of $\widetilde{L}$ by adding the two relations $a = 1, b = 1$. This proves the first assertion.

Assume now that G has cohomological dimension 2. Then L has cohomological dimension at most 2. Consider the E^2 term of the Lyndon-Hochshild-Serre spectral sequence for the extension $1 \to K_P \to \widetilde{L} \to L \to 1$. By cohomological dimension considerations, only the groups $H_p(L, H_q(K_P, \mathbf{Z}))$ for $0 \leq p, q \leq 2$ can be non-zero. Hence at most two of the differentials d^2 can be non-zero maps. One of these is $d^2 : H_2(L, \mathbf{Z}) \to H_0(L, H_1(K_P, \mathbf{Z}))$. Notice that this map has finitely generated cokernel. Moreover, since $E^3 = E^\infty$ the kernel of this map is a section of $H_2(\widetilde{L}, \mathbf{Z})$. Thus, if $H_2(L, \mathbf{Z})$ is not finitely generated, this kernel is not finitely generated and hence $H_2(\widetilde{L}, \mathbf{Z})$ is not finitely generated. This completes the proof.

Theorem 4 *There is a finitely presented group E, given by a presentation which satisfies the $C'(\frac{1}{10})$ small cancellation condition, such that there is no algorithm which determines*

1. *whether an arbitrary finitely generated subgroup of E coincides with E;*
2. *whether an arbitrary finitely generated subgroup of E has finite index;*
3. *whether an arbitrary finitely generated subgroup of E is finitely presented;*
4. *whether an arbitrary finitely generated subgroup S of E has a finitely generated second integral homology group $H_2(S, \mathbf{Z})$;*

Proof: Let P be a finite presentation of the direct product of two free groups of rank $n \geq 2$, say $G = F_n \times F_n$. Observe that G has cohomological dimension 2. In [9] (see also [10] or [8]) a (recursive) collection of finitely generated subgroups $U_i \subseteq G$ for $i = 1, 2, 3, \ldots$ is constructed such that there is no algorithm to determine whether or not $U_i = G$. Each U_i is in fact the pullback of two copies of a map from F_n onto a given finitely presented group. Moreover, from the construction of these subgroups it is clear that if $U_i \neq G$ then U_i has infinite index in G and the generalized word problem for U_i in G is recursively unsolvable.

Since U_i is a pullback as described, if U_i has infinite index in G then by a theorem of Grunewald [7], U_i is not finitely presented. Moreover, if U_i has infinite index in G, Baumslag and Roseblade [3] have shown that $H_2(U_i, \mathbf{Z})$ is not finitely generated. Thus, either $U_i = G$ and is then finitely presented, or $U_i \neq G$ and is then not finitely presented and $H_2(U_i, \mathbf{Z})$ is not finitely generated.

Define $\widetilde{U_i}$ to be the preimages of the U_i in E_P. Then the $\widetilde{U_i}$ are a recursive set of finitely generated subgroups of E_P (given explicitly by their generators). It follows from the construction of the U_i that either $\widetilde{U_i} = E_P$ or $\widetilde{U_i}$ has infinite index in E_P. Hence by Lemma 3, either $\widetilde{U_i} = E_P$ and is then finitely presented, or $\widetilde{U_i} \neq E_P$ and is then not finitely presented and $H_2(\widetilde{U_i}, \mathbf{Z})$ is not finitely generated. Moreover, there is no algorithm to determine for an arbitrary $\widetilde{U_i}$ which of these two possibilities occurs. This completes the proof.

Combining the techniques described above with known unsolvability results for finitely presented groups in general one can similarly prove the following result (see [1], [11] and for a survey [10]): There is a finitely presented group H satisfying the $C'(\frac{1}{10})$ small cancellation condition such that there is no algorithm to determine

1. whether a power of any element of H lies in a fixed finitely generated normal subgroup of H;

2. whether any finitely generated subgroup of H is normal;

3. whether any finitely generated subgroup of H is root-closed;

4. whether any finitely generated normal subgroup of H is a maximal proper normal subgroup;

5. whether any finitely generated subgroup of H has only finitely many conjugates.

We omit the details.

In conclusion, we remark that the kernel K_P in the Rips construction has a number of interesting properties which we plan to explore in a subsequent publication.

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Authors’ addresses:

G. Baumslag
Department of Mathematics
City College of New York
Convent Ave. and 138th Street
New York, N.Y. 10031
USA

C. F. Miller III
Department of Mathematics
University of Melbourne
Parkville, Vic. 3052
Australia

H. Short
Department of Mathematics
City College of New York
Convent Ave. and 138th Street
New York, N.Y. 10031
USA