International Journal of Algebra and Computation
Vol. 19, No. 2 (2009) 213–227

VIRTUAL PROPERTIES OF CYCLICALLY PINCHED ONE-RELATOR GROUPS

GILBERT BAUMSLAG

Department of Computer Science
City College of New York
Convent Ave. and 138th Street
New York, N.Y. 10031, USA
gilbert@groups.sci.ccny.cuny.edu

BENJAMIN FINE

Department of Mathematics and Computer Science
Fairfield University, 1073 North Benson Road
Fairfield CT 06430-5195, USA
fine@mail.fairfield.edu

CHARLES F. MILLER III

Department of Mathematics and Statistics
University of Melbourne
Melbourne 3010, Australia
c.miller@ms.unimelb.edu.au

DOUGLAS TROEGER

Department of Computer Science
City College of New York
Convent Ave. and 138th Street
New York, N.Y. 10031, USA
dtroeger@cs.ccny.cuny.edu

Received 11 September 2008
Revised 28 December 2008

Communicated by J. Meakin

We prove that the amalgamated product of free groups with cyclic amalgamations satisfying certain conditions are virtually free-by-cyclic. In case the cyclic amalgamated subgroups lie outside the derived group such groups are free-by-cyclic. Similarly a one-relator HNN-extension in which the conjugated elements either coincide or are independent modulo the derived group is shown to be free-by-cyclic. In general, the amalgamated product of free groups with cyclic amalgamations is free-by-(torsion-free nilpotent). The special case of the double of a free group amalgamating a cyclic subgroup is shown to be virtually free-by-abelian. Analagous results are obtained for certain one-relator HNN-extensions.

Keywords: One–relator group; cyclically pinched; free-by-cyclic; free product.

2000 Mathematics Subject Classification: 20E06, 20E08, 20E34, 20F05, 20F65

1. Introduction

1.1. *Free subgroups of cyclically pinched one-relator groups*

A group G is termed *a cyclically pinched one-relator group* if it can be expressed as an amalgamated product of the finitely generated free groups A and B with a non-trivial cyclic amalgamation:

$$G = \{A * B;\, a = b\};$$

so here a is a non-trivial element of A and b is a non-trivial element of B [1]. Cyclically pinched one-relator groups arise frequently in the study of groups defined by a single relation. Perhaps the most familiar cyclically pinched one-relator groups are the fundamental groups of both orientable and non-orientable surfaces. In all of these groups the commutator subgroup is free. But this is not true in general. For example, if A is a free on u and v, if B is free on w and x, if $a = [u, v]^2$ and $b = [w, x]^2$, then the derived group of $G = \{A * B; a = b\}$ is not free, as we will see in Sec. 7. (Here we use $[g, h]$ to denote the commutator $g^{-1}h^{-1}gh$ of the elements g and h.) However as we will show this group G contains a subgroup of index 2 which is an extension of a free group by an infinite cyclic group. It is with results of this kind that this paper is concerned.

To explain our focus in more detail we recall some terminology. Suppose that $\mathcal{P}$ is a property or class of groups. Then we say that a group G is *virtually* $\mathcal{P}$ or virtually a $\mathcal{P}$-group, if it contains a subgroup of finite index with the property $\mathcal{P}$. If $\mathcal{Q}$ is a second property of groups, then a group G is said to be $\mathcal{P}$-by-$\mathcal{Q}$ if G contains a normal subgroup K with property $\mathcal{P}$ with G/K a $\mathcal{Q}$-group. The properties that we will consider here include freeness, being abelian and being cyclic.

In this paper we are particularly interested in determining which cyclically-pinched groups are *virtually free-by-cyclic*. Our focus on these groups arises from an old conjecture that one-relator groups with torsion are virtually free-by-cyclic (Baumslag [2]).

In general virtually free-by-cyclic groups turn out to be very well-behaved. For example:

(1) finitely generated virtually free-by-cyclic groups are residually finite [3];
(2) finitely generated virtually free-by-cyclic groups are finitely presented [7].

Somewhat surprisingly, work of Dunfield and Thurston [6] indicates that the preponderance of two-generator one-relator groups are even cyclic extensions of finitely generated free groups.

We now turn our attention to cyclically pinched one-relator groups and to some analogous one-relator groups which are HNN-extensions.

1.2. *Cyclically pinched one-relator groups*

The main objective of this paper is to prove that, given the right conditions, cyclically pinched one-relator groups are virtually extensions of free groups by nice groups. Similar results are proved for certain related HNN-extensions. The connection is given by the well-known observation that the amalgmated product $\{A * B;\, a = b\}$ is embedded in the HNN-extension

$$H = \langle A * B, t;\, t^{-1}at = b\rangle$$

by sending $x \mapsto t^{-1}xt$ for $x \in A$ and $y \mapsto y$ for $y \in B$. Both constructions are instances of the graph of groups construction. We will make use of the subgroup theory for such graphs of groups for the case in which the vertex groups are free and the edge groups are all cyclic.

Our first very general result for these one-relator groups is the following:

Theorem 1. *Suppose that A and B are free groups and $1 \neq a \in A$, $1 \neq b \in B$. Then*

(1) *The cyclically pinched one-relator group*

$$G = \{A * B;\, a = b\}$$

is free-by-(torsion-free nilpotent). Hence G is also free-by-(poly-infinite-cyclic), that is, G contains a free normal subgroup with quotient an iterated extension of cyclic groups.

(2) *The one-relator HNN-extensions*

$$H = \langle A * B, t;\, t^{-1}at = b\rangle \quad \textit{and} \quad L = \langle A, t;\, t^{-1}at = a\rangle$$

are both free-by-(torsion-free nilpotent).

Not all one-relator HNN-extensions are free-by-nilpotent. For example the well known metabelian group $BS(1,2) = \langle a, t;\, t^{-1}at = a^2\rangle$ has no non-abelian free subgroups. Its derived group is isomorphic to the dyadic rational numbers (denominators a power of 2). So it has no normal free subgroups. This example indicates that the sorts of results we are after for a one relator HNN-extension require the elements conjugated by the stable letter to either coincide or be in some way independent.

We are particularly interested in determining which one relator groups are virtually free-by-cyclic or virtually free-by-abelian. In the case a, b do not lie in the derived group the situation is surprisingly positive.

Theorem 2.

(1) *Suppose* $G = \{A * B;\, a = b\}$ *is a cyclically pinched one-relator group. If* $a \notin [A, A]$ *and* $b \notin [B, B]$, *then* G *is free-by-cyclic.*
(2) *Let* F *be a free group and assume that* $a, b \in F$ *but* $a, b \notin [F, F]$. *Suppose that* H *is the one-relator HNN-extension*

$$H = \langle F, t;\, t^{-1}at = b\rangle.$$

If either (i) $a[F, F] = b[F, F]$ *or* (ii) a *and* b *are linearly independent modulo* $[F, F]$, *then* H *is free-by-cyclic.*

Here is a useful special case of part (2) of Theorem 2 obtained by observing that $t^{-1}a^{-1}t = wa^{-1}$ can be written as $[t, a] = w$:

Corollary 3. *Let* $X = \{t, a\} \cup Y$ *be a finite set, with* $a \notin Y, t \notin Y$. *If* w *is a word in the generators* Y, *then*

$$H = \langle X;\, [t, a] = w\rangle$$

is free-by-cyclic. In particular, the fundamental groups of closed orientable surfaces are free-by-cyclic. □

In order to prove our theorems we make use of some general observations about virtual properties and extensions as discussed in the next section. We concentrate on certain classes of virtually free-by-abelian groups and show these are closed under ordinary free products.

1.3. *Doubles*

We also prove two theorems which deal with a special class of cyclically pinched one-relator groups called doubles. We recall that if A is a finitely generated free group, if $\bar{A}$ is an isomorphic copy of A under the isomorphism $x \mapsto \bar{x}$ and if a is a non-trivial element of A, then the amalgamated product

$$G = \{A * \bar{A};\, a = \bar{a}\}$$

is called a *double* of A along $gp(a)$ or simply a double.

A natural HNN-extension analogue of a double is a group of the form $H = \langle A, t;\, t^{-1}at = a\rangle$. In this case the double $G = \{A * \bar{A};\, a = \bar{a}\}$ is embedded in H by the map $x \mapsto x$ for $x \in A$ and $\bar{x} \mapsto txt^{-1}$ for $\bar{x} \in \bar{A}$. Notice that H can also be described as the amalgamated product of A with the free abelian group $\langle s, t;\, t^{-1}st = s\rangle$ amalgamating $gp(s)$ with $gp(a) \leq A$.

Theorem 2 deals with elements which are not in the derived group. Theorem 4 makes a small contribution to the case when they do lie in the derived group, but only in the case of doubles.

Theorem 4. *Let A be finitely generated free group and let u and v be elements of A which are independent modulo $[A, A]$. Suppose that*

(1) $u = u_1^m u_1'$ *where u_1 is not a proper power modulo $[A, A]$ and $u_1' \in [A, A]$;*
(2) $v = v_1^n v_1'$ *where v_1 is not a proper power modulo $[A, A]$ and $v_1' \in [A, A]$;*
(3) $a = [u, v]^e$ *where $e \geq 1$.*

If m and n are odd or if m is odd and $v_1' \in [[A, A], [A, A]]$, then the groups

$$G = \{A * \bar{A};\, a = \bar{a}\} \quad \text{and} \quad H = \langle A, t;\, t^{-1}at = a\rangle$$

are both virtually free-by-cyclic.

Finally we remove any restrictions on the generators of the amalgamated subgroups, but again only in the case of doubles:

Theorem 5. *Doubles of free groups are free-by-virtually-abelian. In more detail: if A is free, then the groups*

$$G = \{A * \bar{A};\, a = \bar{a}\} \quad \text{and} \quad H = \langle A, t;\, t^{-1}at = a\rangle$$

are both free-by-virtually abelian.

As noted earlier, part of this work grew out of an ongoing attempt to prove that one-relator groups with torsion are virtually free-by-cyclic (see also [5]). One of the most interesting examples that came out of this work, which depends on software that we have built to carry out the Reidemeister–Schreier procedure to compute presentations of subgroups of groups given by presentations, is the proof that

$$\langle a, b;\, [a, b^2]^2 = 1\rangle$$

is virtually free-by-cyclic. We will continue this avenue of exploration in [5].

In the last Sec. 7 of this paper, we will illustrate our results with several examples of groups that belong to the various virtual extension classes.

2. Virtual Properties and Extensions

We begin with some general and largely elementary observations about virtual properties and extensions. First recall that a property $\mathcal{P}$ is hereditary if it is subgroup closed, that is, subgroups of $\mathcal{P}$-groups are $\mathcal{P}$-groups.

By definition a group G is virtually $\mathcal{P}$ if it has a subgroup H of finite index which has $\mathcal{P}$. Then H has only finitely many distinct conjugates in G and so their intersection $N = \bigcap_{x \in G} x^{-1}Hx$ is normal and again of finite index. If the property $\mathcal{P}$ is hereditary (or even just inherited by subgroups of finite index), then N will also have $\mathcal{P}$. So in this case G is actually $\mathcal{P}$-by-finite. Of course $\mathcal{P}$-by-finite groups are virtually $\mathcal{P}$. We record this in the following:

Lemma 6. *If property $\mathcal{P}$ is inherited by subgroups of finite index, being virtually $\mathcal{P}$ is equivalent to being $\mathcal{P}$-by-finite.* □

Most of the properties we are concerned with here are hereditary, such as free, abelian, finite, nilpotent, polycyclic, and cyclic. Note that the property "being finitely generated" is inherited by subgroups of finite index, though it is not generally inherited by all subgroups.

Turning our attention to extensions, consider a $\mathcal{P}$-by-$\mathcal{Q}$ group G and suppose that $\mathcal{P}$ and $\mathcal{Q}$ are both hereditary properties. Then G has a normal subgroup N with $N \in \mathcal{P}$ and $G/N \in \mathcal{Q}$. Suppose that H is a subgroup of G. Now the isomorphism theorems tell us that $H \cap N$ is normal in H and $H/H \cap N \cong HN/N$. Since $\mathcal{P}$ is hereditary, $H \cap N$ has property $\mathcal{P}$, and since $\mathcal{Q}$ is hereditary, $H/H \cap N$ has property $\mathcal{Q}$. Thus H is a $\mathcal{P}$-by-$\mathcal{Q}$ group. This shows

Lemma 7. *If $\mathcal{P}$ and $\mathcal{Q}$ are both hereditary properties, then "being a $\mathcal{P}$-by-$\mathcal{Q}$ group" is also hereditary.* □

Two consequences of this discussion are the following:

Corollary 8. *For hereditary properties $\mathcal{P}$ and $\mathcal{Q}$, being virtually $\mathcal{P}$-by-$\mathcal{Q}$ is the same as being $(\mathcal{P}$-by-$\mathcal{Q})$-by-finite.* □

Corollary 9. *The following properties of groups are hereditary:*

(1) *virtually free-by-cyclic;*
(2) *virtually free-by-abelian;*
(3) *free-by-(virtually cyclic).*

One awkwardness that arises is that groups satisfying the weaker condition ($\mathcal{P}$-by-$\mathcal{Q}$)-by-finite, determined by a subnormal series of subgroups, may not be $\mathcal{P}$-by-($\mathcal{Q}$-by-finite) which requires a normal series of subgroups. In more detail, the condition ($\mathcal{P}$-by-$\mathcal{Q}$)-by-finite means there is a subnormal series

$$1 \leq K \leq L \leq G, \tag{1}$$

where $K \in \mathcal{P}$, $L/K \in \mathcal{Q}$ and G/L is finite. Here L is normal in G and K is normal in L but K need not be normal in G. But the condition $\mathcal{P}$-by-($\mathcal{Q}$-by-finite) demands the stronger hypothesis that all the terms of the series are normal in G.

Fortunately there are many situations in which this distinction disappears. We make use of the following particular case:

Lemma 10. *If $\mathcal{P}$ is a hereditary property, then being $(\mathcal{P}$-by-abelian$)$-by-finite is equivalent to being $\mathcal{P}$-by-$($abelian-by-finite$)$. For instance, being virtually free-by-abelian is the same as being free-by-virtually abelian.*

Proof. Since normal series are subnormal, one inclusion is clear. So suppose we have a subnormal series as in (1) where $\mathcal{Q}$ is the class of abelian groups. Since $L/K \in \mathcal{Q}$ is abelian, it follows that the derived group $[L, L]$ of L is contained in K, that is, $[L, L] \leq K$. But $[L, L]$ is characteristic in L (even fully invariant), and hence $[L, L]$ is normal in G. So $1 \leq [L, L] \leq L \leq G$ is a normal series. But $[L, L] \in \mathcal{P}$

since $\mathcal{P}$ is hereditary and $[L, L] \leq K$, and of course $L/[L, L]$ is abelian. Hence this series shows G is $\mathcal{P}$-by-(abelian-by-finite). □

The key aspect of the property "abelian" used in this proof is that it is defined by laws. Hence the maximal abelian quotient of L is determined by the derived group which is the verbal subgroup defined by the laws. The property "finite" of G/L played almost no role. So essentially the same proof can be used to show the following generalization.

Lemma 11. *Let $\mathcal{P}$, $\mathcal{V}$ and $\mathcal{Q}$ be three properties of groups and suppose the $\mathcal{P}$ is hereditary and $\mathcal{V}$ is a variety of groups. Then a group G is a ($\mathcal{P}$-by-$\mathcal{V}$)-by-$\mathcal{Q}$ group if and only if it is a $\mathcal{P}$-by-($\mathcal{V}$-by-$\mathcal{Q}$) group.* □

We have above established that being virtually free-by-abelian is the same as being free-by-virtually abelian. Now we are also particularly interested in the subclass consisting of virtually free-by-cyclic groups. Unfortunately the analog of the above result need not be true for virtually free-by-cyclic groups: such a group need not be free-by-(cyclic-by-finite).

As an example we take G to be the split extension of $L = \mathbb{Z} \times \mathbb{Z}$ by the dihedral group D_4 of order 8 defined by $< s, t;\, s^2 = 1,\ t^2 = 1,\ (st)^4 = 1 >$ with action defined by $(n, m)^s = (-n, m)$ and $(n, m)^t = (m, n)$. Onc can check that the only subgroups invariant under the action are 1 and L, and so L does not contain a proper normal subgroup of G. Now L is free-by-cyclic and so G is (free-by-cyclic)-by-finite but it is not free-by-(cyclic-by-finite).

3. Free Products

Our concern next is with free products of various kinds of virtually free-by-abelian groups.

3.1. *Free products of free-by-virtually-abelian groups*

We begin with the virtually free-by-abelian groups themselves, which we emphasize are the same as the free-by-virtually abelian groups.

Theorem 12. *The free product of two virtually free-by-abelian groups is again virtually free-by-abelian.*

Proof. Let G_i $(i = 1, 2)$ be free-by-virtually-abelian. So G_i has a normal series

$$1 \leq K_i \leq L_i \leq G_i,$$

where K_i is free, L_i/K_i is abelian and L_i is of finite index in G_i. Let

$$P = G_1 * G_2$$

be the free product of the G_i and let

$$D = G_1/K_1 \times G_2/K_2$$

be the direct product of the virtually abelian groups G_i/K_i. Notice that $L_1/K_1 \times L_2/K_2$ is an abelian normal subgroup of D of finite index. Let ϕ be the homomorphism of P onto D mapping G_i canonically onto G_i/K_i $(i = 1, 2)$ and let Q be the kernel of ϕ. Notice that

$$Q \cap G_1 = K_1, \quad Q \cap G_2 = K_2.$$

By the Kurosh subgroup theorem, Q is the a free product of conjugates of subgroups of G_1 and G_2 and a free group. Notice that if one of the factors in this free decomposition for Q is sRs^{-1}, where $R \leq G_1$, then $R \leq Q$. But $Q \cap G_1 \leq K_1$. So $R \leq K_1$. Hence all of the factors in the free product decomposition for Q are free, which means Q is free. Finally observe that

$$P/Q \cong D,$$

and so P is free-by-virtually-abelian. □

3.2. *Free products of free-by-virtually-cyclic groups*

Next we consider free products of groups in the class of free-by-virtually-cyclic groups:

Theorem 13. *The free product of two groups in the class of free-by-virtually-cyclic groups is again in the class of free-by-virtually-cyclic groups.*

We first take care of the case where the factors in the free product are free-by-infinite-cyclic.

Lemma 14. *The free product of free-by-infinite-cyclic groups is free-by-infinite-cyclic.*

Proof. Let P be the free product of the free-by-infinite-cyclic groups P_i where now I denotes an index set and $i \in I$. We can assume that $\mid I \mid > 1$. Let Q_i be a normal subgroup of P_i such that $P_i/Q_i = gp(x_i Q_i)$ is infinite cyclic. Let X be the infinite cyclic group on x and let ψ be the homomorphism of P onto X which maps each of the subgroups Q_i to the identity and x_i to x. We claim that the kernel R of ψ is free and since P/R is infinite cyclic, this will complete the proof of the lemma.

Now by the Kurosh subgroup theorem, R is the free product of a free group and all of the intersections of conjugates of the P_i with R. Let $z_i^{-1} P_i z \cap R$ be one of these intersections. Since R is a normal subgroup of P, this implies that $P_i \cap R$ is one of these intersections. If we now restrict ψ to P_i then we see that ψ maps an arbitrary element ux_i^n $(u \in Q_i)$ to x^n. It follows that $P_i \cap R \leq Q_i$. So the intersections of the conjugates of Q with the P_i are conjugates of the Q_i and are therefore free. So R is free as claimed. □

The rest of the proof of Theorem 13. Let C be the free product of two virtually free-by-infinite-cyclic groups A and B. Let H be a normal subgroup of A of finite index and let I be a normal subgroup of H with H/I infinite cyclic. Similarly, let K be a normal subgroup of B of finite index and let L be a normal subgroup of K with K/L infinite cyclic. Now let ϕ be a homomorphism of the free product $G = A * B$ onto the direct product $A/H \times B/K$ which maps A onto A/H and B onto B/K. Let N be the kernel of ϕ. Then, as in the proof of Lemma 14, N is a free product of a free group and conjugates $y_i^{-1}Ay_i \cap N$ of subgroups of A and conjugates $z_j^{-1}Bz_j \cap N$ of subgroups of B. Consider now one such conjugate, say $N \cap y^{-1}Ay = y^{-1}A_1y$, where A_1 is a subgroup of A. Notice that the subgroup H of A is contained in N and therefore H is contained in A_1. Since the restriction of the homomorphism ϕ to A has kernel H, it follows that we have proved that

$$N \cap y^{-1}Ay = y^{-1}Hy.$$

It follows that we have proved that N is a free product of a free group and conjugates of the subgroups H and K. But these subgroups are free-by-cyclic which means that Lemma 14 applies. This completes the proof of Theorem 13. □

3.3. *Free products of virtually free-by-cyclic groups*

The class of virtually free-by-cyclic groups is also closed under free products:

Theorem 15. *The free product of two virtually free-by-cyclic groups is again a virtually free-by-cyclic group.*

Proof. Our objective now is to prove that if the groups G_i are virtually-free-by-infinite-cyclic ($i = 1, 2$), then so too is their free product G. The proof follows along familiar lines. Thus suppose that

$$1 \leq K_i \leq L_i \leq G_i$$

is a subnormal series of the group G_i ($i = 1, 2$), with L_i of finite index, L_i/K_i cyclic and K_i free. We consider the homomorphism ϕ from G onto the finite group $G_1/L_1 \times G_2/L_2$ defined by mapping G_1 canonically onto G_1/L_1 and G_2 onto G_2/L_2. Let Q be the kernel of ϕ. Then again by the Kurosh subgroup theorem Q is a free product of conjugates of subgroups of the L_i and a free group. Since $L_i \leq Q$, it follows that these intersections are conjugates of L_1 and L_2. Therefore Q is a free product of free-by-infinite-cyclic groups and so Q is also free-infinite-cyclic by Lemma 14. This completes the proof of Theorem 15. □

4. The proofs of Theorems 1 and 2

We come next to the proof of Theorem 1. Although the proof is not difficult, it requires some preparation.

4.1. Central products

We recall first the definition. To this end, let X and Y be an arbitrary pair of groups and let H be a subgroup of the center of X and let K be a subgroup of the center of Y. Suppose now that ϕ is an isomorphism from H to K. Furthermore, let

$$D = X \times Y$$

be the direct product of X and Y and let

$$J = \{(h, h^{-1}\phi) \mid h \in H\}.$$

Then J is a subgroup of the center of D and hence normal. The quotient group $C = D/J$ is termed the *central* product of X and Y with H identified with K according to the ismorphism ϕ. It is easy to check that X and Y are embedded in C. If we identify X and Y with their images in C, then we find that

$$H = X \cap Y = K.$$

We will need some simple observations about these central products. The verification of these are straightforward and are left to the reader.

Lemma 16. *Let C be the central product of the groups X and Y with H and K identified. Then the following hold*:

(1) *If x is an element of infinite order in X, then the image of x in C is of infinite order.*
(2) *If X and Y are nilpotent, then so is C.*
(3) *If X and Y are nilpotent, if x is an element of infinite order in X and T is the torsion subgroup of C, then xT is an element of infinite order in the torsion-free nilpotent group C/T.*

4.2. The proof of Theorem 1

First we consider part (1). Recall that free groups are residually torsion-free nilpotent. Hence there exist normal subgroups M of A and N of B such that

(1) A/M and B/N are torsion-free nilpotent;
(2) $a \notin M$ and $b \notin N$;
(3) aM is in the center of A/M and bN is in the center of B/N.

Let

$$C = \{A/M \times B/N;\, aM = bN\}$$

be the central product of A/M and B/N amalgamating aM with bN and let T be the torsion subgroup of C. Then it follows from Lemma 16 that C/T is a finitely generated, torsion-free nilpotent group and hence poly-infinite-cyclic.

Now let ϕ be the homomorphism of G onto C/T defined as follows. It is the extension of the canonical homomorphism of A onto A/M followed by the canonical homomorphism into C/T and that of B onto B/N followed by the canonical homomorphism into C/T. Observe that ϕ is monic on the amalgamated subgroup (generated by a). Hence, by a theorem of Hanna Neumann [9], the kernel K of ϕ is the intersection of conjugates of subgroups of A, and B and a free group. So K is free and this then completes the proof part (1) of the theorem.

For part (2) of Theorem 1 we proceed in the same manner: Let M, N and ϕ be as in part (1) of the proof. Map $H = \langle A * B, t;\, t^{-1}at = b\rangle$ onto G by sending $t \mapsto 1$ and then compose with ϕ to obtain a homomorphism $\theta : H \to C/T$ which is monic on $gp(a)$. As in part (1), the subgroup theory for a graph of groups implies the kernel of ϕ is free.

For L we again find a term $M = \gamma_{c+1}(A)$ of the lower central series of A so that $a \notin M$ and aM is in the center of A/M. The map ψ from $L = \langle A, t;\, t^{-1}at = a\rangle$ to A/M which sends $a \mapsto aM$ and $t \mapsto 1$ is monic on $gp(a)$. So by the analogue of Hanna Neumann's theorem for graphs of groups, the kernel of ψ is free. This completes the proof of part (2) and hence the theorem.

4.3. *The proof of Theorem 2*

We consider the two parts of the theorem separately dealing first with the cyclically pinched case.

Theorem 17. *Suppose $G = \{A * B;\, a = b\}$ is a cyclically pinched one-relator group. If $a \notin [A, A]$ and $b \notin [B, B]$, then G is free-by-cyclic.*

Proof. Suppose that the generators of A are $x_1, x_2, \ldots, x_n$ and the generators of B are $y_1, y_2, \ldots, y_m$. If w is any word, we denote the exponent sum of the occurrences of the letter x_i in w by $\sigma(x_i, w)$, and similarly for $\sigma(y_j, w)$. Looking at the word a, if any $\sigma(x_i, a) < 0$, we replace the generator x_i by x_i^{-1} so that in the revised generating set we have $\sigma(x_i, a) \geq 0$. Similarly we may assume $\sigma(y_j, b) \geq 0$ for each $j = 1, \ldots, m$.

Now define

$$\alpha = \sum_{i=1}^{n} \sigma(x_i, a) \quad \text{and} \quad \beta = \sum_{j=1}^{m} \sigma(y_j, b)$$

which are the total exponent sums on a and b respectively. Note that both α and β are positive since $a \notin [A, A]$ and $b \notin [B, B]$ because of our adjustment of generators.

We now define the map ϕ from the generators of G to the infinite cyclic group $\mathbb{Z}$ by $x_i \mapsto \beta$ and $y_j \mapsto \alpha$. Extending this map formally to all words, one can check that $\phi(a) = \alpha\beta > 0$ and $\phi(b) = \beta\alpha = \phi(a)$. Hence ϕ defines a homomorphism from G onto an infinite cyclic subgroup of $\mathbb{Z}$.

Since $\phi(a) = \phi(b) = \alpha\beta > 0$, the kernel of ϕ does not intersect any conjugate of $gp(a) = gp(b)$. Since A and B are both free, Hanna Neumann's Theorem implies that the kernel of ϕ is free. Hence G is free-by-cyclic as claimed. □

Next we want to prove a related result for one relator groups given as HNN-extensions. The argument is a generalization of the above proof and makes use of the following lemma.

Lemma 18. *Suppose R is an integral domain and R^n is the free R-module of rank n. Suppose $0 \neq a, b \in R^n$. Then there is an R-homomorphism $\phi : R^n \to R$ with $\phi(a) = \phi(b) \neq 0$ if and only if either $a = b$ or a and b are R-linearly independent.*

Proof. Suppose first that R is a field. If a and b are R-linearly independent then they can be included in a basis. Sending each basis element to $1 \in R$ gives the desired linear map. The rest of the assertion is clear. In the more general case that R is an integral domain, we know the result holds for its field of fractions. Redefining the map ϕ to send the basis elements to a suitable $0 \neq r \in R$ so as to clear the fractions again gives the desired R-linear map. Again the rest is clear. □

Theorem 19. *Let F be a free group and assume that $a, b \in F$ but $a, b \notin [F, F]$. Suppose that H is the one-relator HNN-extension*

$$H = \langle F, t;\, t^{-1}at = b\rangle.$$

If either (i) $a[F, F] = b[F, F]$ *or* (ii) *a and b are linearly independent modulo $[F, F]$, then H is free-by-cyclic.*

Proof. Applying the Lemma 18 for the case $F/[F, F] \cong \mathbb{Z}^n$, there is a $\mathbb{Z}$-homomorphism $\phi : F/F \to \mathbb{Z}$ such that $\phi(a[F, F]) = \phi(b[F, F]) \neq 0$. Composing ϕ with the quotient map $F \to F/[F, F]$ gives a homomorphism $\psi : F \to \mathbb{Z}$ with $\psi(a) = \psi(b) \neq 0$. Then the map $\theta : H \to \mathbb{Z}$ defined by $\theta(u) = \psi(u)$ for $u \in F$ and $\theta(t) = 0$ naturally extends to a homomorphism.

Since $\theta(a) = \theta(b) \neq 0$ the kernel of θ intersects the conjugates of $gp(a)$ and $gp(b)$ trivially. By the subgroup theory for graphs of groups (which generalizes Hanna Neumann's theorem), the kernel of θ is a graph of groups with vertex groups which are conjugates of subgroups of F and whose vertex groups are trivial. Hence the kernel of θ is free. This proves the theorem and so also completes the proof of Theorem 2. □

Although we have chosen to prove these results separately by similar arguments, one can alternatively deduce Theorem 17 directly from Theorem 19. To see this we observe that $G = \{A * B;\, a = b\}$ can be embedded in $H = \langle A * B, t;\, t^{-1}at = b\rangle$ by sending $x \mapsto t^{-1}xt$ for $x \in A$ and $y \mapsto y$ for $y \in B$. Under the hypothesis of Theorem 1, a and b are clearly linearly independent modulo the derived group of $A * B$. Thus H is free-by-cyclic and hence $\{A * B;\, a = b\}$ is free-by-cyclic because the property is hereditary.

5. The Proof of Theorem 4

Since being virtually free-by-cyclic is a hereditary property and G is embedded in $H = \langle A, t;\, t^{-1}at = a\rangle$ it suffices to show that H is virtually free by cyclic. The proof depends on concocting a homomorphism ϕ of A onto the infinite dihedral group so that image of a is of infinite order. To this end, let

$$D = \langle c, d;\ c^2 = 1,\ cdc = d^{-1} \rangle$$

be the infinite dihedral group. We can assume after transforming the free generators of A by an appropriate automorphism, that u_1 and v_1 are part of a free basis of A. Let $\phi : G \to D$ be the homomorphism of H onto D defined by mapping all of the generators of H to the identity except for the generators u_1, v_1, t, which are mapped into D as follows:

$$u_1\phi = c, \quad v_1\phi = d, \quad t\phi = 1.$$

Now the derived group of D is generated by d^2. It follows that

$$a\phi = [u_1^m u_1', v_1^n v_1']^e \phi = [c^m d^{2j}, d^n d^{2k}]^e$$

for an appropriate choice of the integers j and k. Now if m and n are odd it follows that $n + 2k \neq 0$ and hence that

$$a\phi = [c, d^{n+2k}]^e = d^{2(n+2k)e} \neq 1.$$

On the other hand, if m is odd and $v_1' \in [[A, A], [A, A]]$, then the integer $k = 0$ and

$$a\phi = [u_1^m u_1', v_1^n v_1']^e \phi = [c^m d^{2j}, d^n]^e = d^{2ne} \neq 1.$$

Now if Q is the kernel of ϕ, then Q intersects the subgroup generated by a in the identity. So Q is free as usual. Moreover the image of ϕ is an infinite subgroup of the infinite dihedral group and hence is virtually infinite cyclic. This completes the proof of Theorem 4.

6. The General Case of Doubles

Since being free-by-virtually abelian is a hereditary property and G is embedded in $H = \langle A, t;\ t^{-1}at = a \rangle$ it suffices to show that H is free-by-virtually abelian. Now by a theorem of Marshall Hall [8], a finitely generated subgroup of a free group is a free factor of a subgroup of finite index. So there is a subgroup V of A of finite index which is a free product of the subgroup $gp(a)$ generated by a and a subgroup W of V:

$$V = gp(a) * W.$$

Let X be the intersection of the conjugates of V in A. Then X is a normal subgroup of A of finite index. So aX is an element of finite order in A/X. Thus if e is the order of a modulo X, $a^e \in X$ and generates a free factor of X:

$$X = gp(a^e) * Y.$$

It follows that a^e is of infinite order modulo $[X, X]$, the derived group of X. Moreover $[X, X]$ is characteristic in the normal subgroup X of A and hence is normal in A. Also observe that $A/[X, X]$ is virtually abelian and $a[X, X]$ is of infinite order in $A/[X, X]$.

Now we define a homomorphism ϕ of H onto $A/[X, X]$ which is the canonical homomorphism of A onto $A/[X, X]$ and sends t to the identity. The kernel Q of ϕ intersects $gp(a)$ trivially because the image of a in $A/[X, X]$ is of infinite order and Q is therefore free. Since $H/Q \cong A/[X, X]$, it follows that we have proved that G is free-by-virtually-abelian, as desired.

7. Examples

We describe next some illustrations of our theorems (see also [4]).

To begin with we have the following consequence of Theorem 2.

Example 1. If $e_1, \ldots, e_m, f_1, \ldots, f_n, i, j$ are nonzero integers, then

$$G = \langle a_1, \ldots, a_m, b_1, \ldots, b_n;\ (a_1^{e_1} \cdots a_n^{e_m})^i = (b_1^{f_1} \cdots b_n^{f_n})^j \rangle$$

is free-by-cyclic.

So, in particular, the familiar examples

$$G(\ell, m, n) = \langle a, b, c;\ a^\ell b^m c^n \rangle$$

where ℓ, m, n are nonzero integers, are free-by-cyclic. However, not all cyclically pinched one-relator groups are virtually free-by-cyclic:

Example 2. The groups

$$G = \langle a, b, c, d;\ [a, b]^e = [c, d]^e \rangle$$

where $e \geq 1$ are free-by-cyclic if and only if $e = 1$.

Indeed, observe that $[G, G]$ contains the elements $[a, b]$ and $[c, d]$. Now in a free group elements with equal powers are equal. But if $e > 1$, $[a, b] \neq [c, d]$ in G. So if $e > 1$, G is not free-by-cyclic, as claimed. However, it follows from Theorem 4, that G is virtually free-by-cyclic, no matter the value of e. When $e = 1$, G is the fundamental group of a two-dimensional orientable surface and, as is well-known, G is free-by-cyclic.

Finally, as an instance of Corollary 3, we have:

Example 3. The groups

$$\langle x, y, z;\ x^n = [y, z] \rangle$$

are free by cyclic.

The above examples have all been the subject of investigations in a variety of contexts.

References

[1] G. Baumslag, On generalised free products, *Math. Z.* **78** (1962) 423–438.
[2] G. Baumslag, Residually finite one-relator groups, *Bull. Amer. Math. Soc.* **73** (1967) 618–620.
[3] G. Baumslag, Finitely generated cyclic extensions of free groups are residually finite, *Bull. Austral. Math. Soc.* **5** (1971) 87–94.
[4] G. Baumslag, Free subgroups of certain one-relator groups defined by positive words, *Math. Proc. Camb. Phi. Soc.* **93** (1983) 247–251.
[5] G. Baumslag and D. R. Troeger, One-relator groups with torsion, in preparation.
[6] N. Dunfield and D. Thurston, Finite covers of random 3-manifolds, *Geom. Topol.* **166** (2006) 457–521.
[7] M. Feighn and M. Handel, Mapping tori of free group automorphisms are coherent, *Ann. Math.* **2** **149**(3) (1999) 1061–1077.
[8] M. Hall, Subgroups of finite index in free groups, *Canad. J. Math.* **1** (1949) 421–432.
[9] H. Neumann, Generalised free products with amalgamated subgroups, I. II *Amer. J. Math.* **70** (1948) 690–625; **71** (1949) 491–540.