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Some Remarks about Multiplicators and Finitely Presented Groups

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1. Introduction

Let G be a finitely generated group. If G is finitely related, then it is easy to prove that the (Schur-)multiplicator $m(G)$ of G is finitely generated (see e.g. either [5] or [4]). It was conjectured for some time that the converse is false. This has recently been verified [2]. The question as to how the homology of a group mirrors its presentations, however, is really still unanswered. Here we offer two facts, which are somewhat at variance with each other. The first is an almost obvious positive result.

Theorem A. *Let W_p be the standard wreath product of a group of prime order p by an infinite cyclic group, and let G be a finitely generated metabelian group. If W_p is a factor group of G then the multiplicator of G is not finitely generated.*

The second fact is the antithesis of the first.

Theorem B. *Let p be an odd prime which is not divisible by 3. Then there exists a 3-generator, metabelian extension G_p of W_p by a cyclic group of order $p-1$ which is not finitely related although its multiplicator is cyclic (and hence finitely generated).*

We prove Theorem A in **2.** and Theorem B in **3.**

2. The Proof of Theorem A

The proof of the theorem depends on the following three easily proved lemmas.

Lemma 1. *Let G be a finitely generated group. Then $m(G)$ is finitely generated if and only if the left-hand end of every finitely generated central extension by G is finitely generated; that is, if*

$$1 \rightarrow C \rightarrow E \rightarrow G \rightarrow 1$$

is any short exact sequence where C is a central subgroup of E and E is finitely generated, then C is finitely generated.

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This lemma is well-known and so the proof is omitted.

Lemma 2. *Let*

$$1 \to C \to E \to G \to 1$$

be a central extension. If G is a finitely generated metabelian group and C is finitely generated, then $D/[D, E]$ is finitely generated for every normal subgroup D of E.

Proof. A finitely generated metabelian group satisfies the maximal condition for normal subgroups (Hall [6]). Hence D is finitely generated as a normal subgroup of E modulo C. Since C is a finitely generated central subgroup of G it follows readily that $D/[D, E]$ too is finitely generated.

Lemma 3. *Let G be a finitely generated metabelian group. Then $m(G)$ is finitely generated if, and only if, $m(H)$ is finitely generated for every homomorphic image H of G.*

Proof. We present G as a factor group of a finitely generated free group F i.e. $G \cong F/R$. By Lemma 1 $m(G)$ is finitely generated if, and only if, $R/[F, R]$ is finitely generated. Now any homomorphic image H of G can be presented in the form $H \cong F/S$ where $S \geqq R$. It follows from Lemma 2 that if $R/[F, R]$ is finitely generated, then $S/[F, S]$ is also finitely generated. So if the multiplicator of G is finitely generated, then the multiplicator of every homomorphic image of G is also finitely generated. This completes the proof of Lemma 3.

Theorem A follows easily. It is enough, by Lemma 3, to prove that $m(W_p)$ is not finitely generated. It therefore suffices, by Lemma 1, to construct a central extension

$$1 \to C \to E \to W_p \to 1$$

in which E is finitely generated but C is not. Such central extensions are plentiful (see Hall [6], pp. 434, 435). This completes the proof of Theorem A.

There is an easy consequence of Theorem A that is, perhaps, worth mentioning.

Corollary. *Let F be a finitely generated free group and let R be a normal subgroup of F. If F/R is an infinite abelian group, then $m(F/[R, R])$ is not finitely generated.*

It follows from this corollary that the wreath product $P = A \wr B$ of two finitely generated abelian groups A and B has a finitely generated multiplicator, if, and only if, either $A = 1$ or B is finite. Therefore P is finitely presented if, and only if, either $A = 1$ or B is finite. This is a special case of the main result in [1].

3. The Proof of Theorem B

Let p be an odd prime which is not divisible by 3. We shall construct a 3-generator metabelian extension G_p of W_p (see Theorem A) by a cyclic group of order $p-1$ which is not finitely related although its multiplicator is cyclic (and hence finitely generated).

To this end let B be a countably infinite abelian group of exponent p:

$$B = \langle \ldots, b_{-1}, b_0, b_1, \ldots; [b_i, b_j] = b_i^p = 1, i, j = 0, \pm 1, \ldots \rangle.$$

Then G_p is a split extension of B by the direct product of a cyclic group of order $p-1$ on s and the infinite cyclic group on t, where the action of s and t on B is described in the following presentation:

$$G_p = \langle \ldots, b_{-1}, b_0, b_1, \ldots, s, t; [s, t] = s^{p-1} = [b_i, b_j] = b_i^p = 1, b_i^s = b_i^2, \\ b_i^t = b_{i+1}, i, j = 0, \pm 1, \ldots \rangle.$$

On putting $b = b_0$, we find that $G_p = gp(b, s, t)$. The above presentation for G_p can now be rewritten in terms of these generators as follows:

$$G_p = \langle b, s, t; s^{p-1} = [s, t] = 1, b^s = b^2, b^p = [b, t^{-i} b t^i] = 1, i = 0, \pm 1, \ldots \rangle.$$

Let us put $W_p = gp(b, t)$. Then W_p is the wreath product of a group of order p by an infinite cyclic group. Therefore W_p is not finitely presented (e.g. by Theorem A). Now W_p is of finite index $p-1$ in G_p. So G_p is not finitely presented.

Our objective now is to prove that G_p has a cyclic multiplicator. To do so, let us consider the free group F on β, σ, τ and let R be the normal subgroup of F generated by the elements

$$[\sigma, \tau], \beta^p, \sigma^{p-1}, [\beta, \tau^{-i} \beta \tau^i], \quad i = 0, \pm 1, \ldots, \sigma^{-1} \beta \sigma \beta^{-2}.$$

Then $m(G_p) = [F, F] \cap R/[F, R]$. Put

$$\bar{\sigma} = \sigma[F, R], \quad \bar{\tau} = \tau[F, R], \quad \bar{\beta} = \beta[F, R], \quad \bar{F} = F/[F, R], \quad \bar{R} = R/[F, R].$$

Furthermore, we put $\bar{\beta}_i = \bar{\tau}^{-i} \bar{\beta} \bar{\tau}^i$ $(i = 0, \pm 1, \ldots)$.

Now $\bar{R}$ is central in $\bar{F}$. Hence, for each i,

$$[\bar{\beta}_0, \bar{\beta}_i] = \bar{\sigma}^{-1} [\bar{\beta}_0, \bar{\beta}_i] \bar{\sigma} = [\bar{\sigma}^{-1} \bar{\beta}_0 \bar{\sigma}, \bar{\sigma}^{-1} \bar{\beta}_i \bar{\sigma}].$$

Now $\bar{\sigma}^{-1} \bar{\beta}_i \bar{\sigma} = \bar{\beta}_i^2 \bar{r}_i$ for an appropriate choice of $\bar{r}_i \in \bar{R}$. Hence, remembering that $\bar{R}$ is central in $\bar{F}$, we find that

$$[\bar{\beta}_0, \bar{\beta}_i] = [\bar{\beta}_0^2 \bar{r}_0, \bar{\beta}_i^2 \bar{r}_i] = [\bar{\beta}_0^2, \bar{\beta}_i^2] = [\bar{\beta}_0, \bar{\beta}_i]^4.$$

This means that

$$[\bar{\beta}_0, \bar{\beta}_i]^3 = 1.$$

But because $\bar{\beta}_0^p \in \bar{R}$, we find

$$1 = [\bar{\beta}_0^p, \bar{\beta}_i] = [\bar{\beta}_0, \bar{\beta}_i]^p.$$

Since p is not divisible by 3 these two conditions yield

$$[\bar{\beta}_0, \bar{\beta}_i] = 1 \qquad \text{for } i = 0, \pm 1, \ldots.$$

Therefore $\bar{R} = gp([\bar{\sigma}, \bar{\tau}], \bar{\beta}^p, \bar{\sigma}^{p-1}, \bar{\sigma}^{-1} \bar{\beta} \bar{\sigma} \bar{\beta}^{-2})$. It follows directly from this fact that

$$[\bar{F}, \bar{F}] \cap \bar{R} = gp([\bar{\sigma}, \bar{\tau}]),$$

and hence $m(G_p)$ is cyclic as claimed. This completes the proof of Theorem B.

It is worth noting that although $m(G_p)$ is cyclic, $m(W_p)$ is not even finitely generated (by Theorem A), even though W_p is of finite index in G_p.

It is also worth pointing out that by varying the construction slightly it is possible to construct a torsion-free metabelian group, whose derived group is of infinite rank, which is not finitely related but nevertheless has a finitely generated multiplicator (cf. also [3]).

References

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