Journal of Pure and Applied Algebra 8 (1976) 187–196

SOME REMARKS ON FINITELY PRESENTED ALGEBRAS

Gilbert BAUMSLAG*
City College of the City University of New York, Convent Ave. at 138th St., New York, N.Y. 10031, U.S.A.

Communicated by K.W. Gruenberg
Received 25 September 1975

1. The theorems

In this note we shall prove three theorems concerned with the presentability of certain associative and Lie algebras.
First we have the simple

Theorem A. *Let G be a group, k a field, Then the group algebra kG of G over k is finitely presented if and only if G is finitely presented.*

Second we have the pleasing

Theorem B. *Let L be a Lie-algebra over the field k and let U be its universal enveloping algebra. Then L is finitely presented if and only if U is finitely presented.*

Finally we offer an example of a finitely generated Lie algebra which shares with finitely presented Lie algebras a homological property without itself being finitely presented.

Theorem C. *There exists a finitely generated rational Lie algebra L which is not finitely related but whose second homology group $H_2(L, \mathbf{Q})$ (with coefficients in the additive group of rational numbers $\mathbf{Q}$ viewed as a trivial L-module) is finite dimensional.*

We shall explain all of the terminology used here, as well as the notation, in Section 2; and prove and comment on Theorems A, B and C in Sections 3, 4 and 5 respectively.

* Support from the National Science Foundation is gratefully acknowledged.

2. Preliminaries

2.1. As usual an algebra over a field k is a left vector space A over k equipped with a binary operation $(a, b) \to ab$ satisfying the identity $\alpha(ab) = (\alpha a)b = a(\alpha b)$ for all $\alpha \in k$, $a, b \in A$ and the distributive laws. A is termed associative if it satisfies the associative law; similarly if A satisfies the Lie identities

$$a^2 = 0, \ (ab)c + (bc)a + (ca)b = 0 \ \ (a, b, c \in A)$$

it is termed a Lie algebra. Free associative and free Lie algebras are defined in the customary way; so are the notions finitely generated, finitely presented associative algebra, finitely presented Lie algebra (see e.g. Cohn [1, p. 153]).

2.2. If A is any associative algebra, then A may be turned into a Lie algebra $(A, \circ)$ by using the same vector space structure together with a new binary operation $\circ$ defined by

$$(a, b) \to a \circ b = ab - ba \ \ (a, b \in A).$$

$(A, \circ)$ is termed the *commutation Lie algebra* of (the associative algebra) A. If L is any Lie algebra, then there exists a unique (up to isomorphism) associative algebra U, termed the universal enveloping algebra of L, such that $(U, \circ)$ contains an isomorphic copy of L (for a precise definition of U as well as a detailed discussion of its properties see Jacobsen [2, Chapter 5]).

2.3. Let now L be a Lie algebra and suppose L is presented as a factor algebra of a free Lie algebra F:

$$L \cong F/R.$$

If F^2 is the subalgebra of F generated by the products $ab (a, b \in F)$ and if, similarly, FR is the subalgebra generated by the products $ab (a \in F, \ b \in R)$, then it turns out that the abelian Lie algebra

$$\frac{F^2 \cap R}{FR}$$

is independent of the choice of the presentation for L in the sense that it is isomorphic to $H_2(L, k)$, where k is the underlying ground field (Knopfmacher [3]). Notice that an abelian Lie algebra is simply a vector space with the zero multiplication and hence is completely determined by its dimension.

3. The proof of Theorem A

3.1. We begin the proof of Theorem A with the following probably well-known

Lemma A1. *Let G be a group, k a field. Then kG is finitely generated (as an associative algebra) if and only if G is a finitely generated group.*

If G is finitely generated, then clearly so too is kG.
On the other hand suppose G is not finitely generated. If G is uncountable, kG has uncountable dimension and so cannot be finitely generated. Therefore we may assume that G is countable and hence can be written as a properly ascending infinite union of its subgroups G_i:

$$G = \bigcup_{i=1}^{\infty} G_i.$$

It follows that kG is an infinite properly ascending union of its subalgebras kG_i:

$$kG = \bigcup_{i=1}^{\infty} kG_i.$$

Therefore kG is not finitely generated, as required.

3.2. Next we prove

Lemma A2. *Let G be a finitely generated but not finitely related group. Then kG is not finitely presented (as an associative algebra).*

Proof. Let F be a finitely generated free group with free basis $f_1, f_2, ..., f_n$ which maps onto G with kernel R:

$$F/R \cong G.$$

Since G is not finitely related we may express R as an infinite properly ascending union of normal subgroups R_i of F:

$$R = \bigcup_{i=1}^{\infty} R_i.$$

The natural homomorphism of F onto F/R_i induces a homomorphism of kF onto $k(F/R_i)$ with kernel J_i, say. It follows that

$$(1 - J_i) \cap R = R_i$$

where $1 - J_i = \{1 - a \mid a \in J_i\}$. Consequently

$$J = \bigcup_{i=1}^{\infty} J_i$$

is a properly ascending infinite union of the ideals J_i and hence is not finitely generated as an ideal. Notice that

$$kF/J \cong kG.$$

Now let A be the free associative algebra (without 1) on $a_1, b_1, ..., a_n, b_n, a_{n+1}$ and let φ be the homomorphism of A onto kF defined by

$$a_1 \to f_1,\ b_1 \to f_1^{-1}, ..., a_n \to f_n,\ b_n \to f_n^{-1},\ a_{n+1} \to 1.$$

If $K = \varphi^{-1}(J)$ then K is not finitely generated as an ideal of A because $K\varphi = J$ and J is not finitely generated as an ideal of kF. Since

$$A/K \cong kG$$

it follows that kG is not a finitely related associative algebra, as desired (see Cohn [1, p. 154]).

3.3. It is easy now to prove

Theorem A. *Let G be a group, k a field. Then the group algebra kG of G over k is finitely presented if and only if G is finitely presented.*

Proof. It is clear that kG is finitely presented if G is finitely presented. Conversely suppose kG is finitely presented. Then G is finitely generated by Lemma A1 and by Lemma A2 G is also finitely related. So G is finitely presented. This completes the proof of Theorem A.

4. The proof of Theorem B

4.1. Let L be a Lie algebra and let U be its universal enveloping algebra. As we have already noted (in Section 2) L may be identified with a Lie subalgebra of the Lie algebra $(U, \circ)$ which we shall again denote by L. Now the associative subalgebra of U generated by L is U (see Jacobson [2, p. 162]). So if the Lie algebra L is finitely generated, so too is its universal enveloping algebra. Thus we have proved one half of

Lemma B1. *The Lie algebra L is finitely generated if and only if its universal enveloping algebra U is finitely generated.*

Proof. It is clear that if the dimension $\dim L$ of L is uncountable then so too is $\dim U$. Therefore U is certainly not finitely generated if $\dim L$ is uncountable. Suppose then that $\dim L$ is countable, but that L is not finitely generated. Then L can be written as an infinite properly ascending union of its Lie subalgebras L_i:

$$L = \bigcup_{i=1}^{\infty} L_i.$$

Let U_i be the associative subalgebra of U generated by L_i. Then (see Jacobson [2, p. 162])

$$U_i \cap L = L_i.$$

Hence

$$U = \bigcup_{i=1}^{\infty} U_i$$

is an infinite properly ascending union of its subalgebras U_i and consequently is not finitely generated. This completes the proof of Lemma B1.

4.2. We prove next the analogue of Lemma A2.

Lemma B2. *Let L be a finitely generated but not finitely related Lie algebra. Then its universal enveloping algebra U is not finitely related.*

Proof. Let M be a finitely generated free Lie algebra which maps onto L with kernel K, say:

$$M/K \cong L.$$

Since L is not finitely related we can express K as an infinite properly ascending union of ideals K_i of M:

$$K = \bigcup_{i=1}^{\infty} K_i.$$

Let now V be the universal enveloping algebra of M. Then V is a finitely generated free associative algebra. As usual we identify M with a Lie subalgebra of $(V, \circ)$; so it makes sense to define T_i to be the ideal of the associative algebra V generated by K_i and T to be the ideal of V generated by K. It is clear then that

$$T = \bigcup_{i=1}^{\infty} T_i.$$

Now for every i

$$T_i \cap M = K_i$$

(see Jacobson [2, p. 162]). So T is an infinite union of a properly ascending chain of ideals of V of type ω and hence T is not a finitely generated ideal. Consequently V/T is not finitely related. But (again by Jacobson [2, p. 162])

$$V/T \cong U$$

which means U is not finitely related, as claimed.

4.3. The proof of Theorem B follows easily now.

Theorem B. *Let L be a Lie algebra over the field k and let U be its universal enveloping algebra. Then L is finitely presented if and only if U is finitely presented.*

Proof. Suppose L is finitely presented. Then it is clear that U is also finitely presented (see Jacobson [2, Chapter 5]).

On the other hand suppose U is finitely presented. Then by Lemma B1 L is finitely generated and by Lemma B2 L is then finitely presented. This completes the proof.

5. The proof of Theorem C

5.1. Let L be the Lie algebra over $\mathbf{Q}$ defined as follows:

$$L = \langle a, s, t; a((\ldots\underbrace{(at)\ldots)}_{i}t) = 0\ (i = 1, 2, \ldots),\ st = 0,\ as = 2a\rangle .$$

Our objective in this section is to prove $H_2(L, \mathbf{Q})$ is finite dimensional. In Section 5.2 we shall prove that L is not finitely presented. Thus L is a finitely generated Lie algebra over $\mathbf{Q}$ with $H_2(L, \mathbf{Q})$ finite dimensional although it is not finitely related, which proves Theorem C.

First of all then we have

Lemma C1. *$H_2(L, \mathbf{Q})$ is finite dimensional.*

Proof. Let F be the free Lie algebra on α, σ and τ and let R be the kernel of the homomorphism of F onto L defined by $\alpha \to a$, $\sigma \to s$, $\tau \to t$. Then

$$L \cong F/R$$

and R is generated, as an ideal, by the elements

$$(1) \qquad \alpha((\ldots\underbrace{(\alpha\tau)\ldots)}_{i}\tau)\ (i = 1, 2, \ldots),\ \sigma\tau,\ \alpha\sigma - 2\alpha.$$

Our objective is to prove

$$H_2(L, \mathbf{Q})\ (\cong F^2 \cap R/FR) \text{ finite dimensional.}$$

To this end put

$$\bar{f} = f + FR\ (f \in F),\ \bar{F} = F/FR,\ \bar{R} = R/FR.$$

Then it follows from (1) that

$$(2) \qquad \bar{R} = \mathcal{L}a\,(\bar{\alpha}((\ldots\underbrace{(\bar{\alpha}\bar{\tau})\ldots)}_{i}\bar{\tau})\ (i = 1, 2, \ldots),\ \bar{\sigma}\bar{\tau},\ \bar{\alpha}\bar{\sigma} - 2\bar{\alpha}),$$

i.e. $\bar{R}$ is the Lie subalgebra of $\bar{F}$ generated by the exhibited elements. Notice that if $\bar{r} \in \bar{R}$, then $\bar{r}\bar{f} = 0$ for every $\bar{f} \in \bar{F}$. Now put

$$\bar{\alpha}_i = (\ldots\underbrace{(\bar{\alpha}\bar{\tau})\ldots)}_{i}\bar{\tau}\ \ (i = 1, 2, \ldots).$$

Then

$$\bar{\alpha}_1\bar{\sigma} = (\bar{\alpha}\bar{\tau})\bar{\sigma} = -(\bar{\tau}\bar{\sigma})\bar{\alpha} - (\bar{\sigma}\bar{\alpha})\bar{\tau} = (\bar{\alpha}\bar{\sigma})\bar{\tau} = ((\bar{\alpha}\bar{\sigma} - 2\bar{\alpha}) + 2\bar{\alpha})\bar{\tau} = 2\bar{\alpha}_1 .$$

It follows inductively that

$$\bar{\alpha}_i\bar{\sigma} = 2\bar{\alpha}_i\ (i = 1, 2, \ldots).$$

Hence

$$0 = (\bar{\alpha}\bar{\alpha}_i)\bar{\sigma} = -(\bar{\alpha}_i\bar{\sigma})\bar{\alpha} - (\bar{\sigma}\bar{\alpha})\bar{\alpha}_i = -2\bar{\alpha}_i\bar{\alpha} + (\bar{\alpha}\bar{\sigma})\bar{\alpha}_i .$$

Therefore

$$0 = 2\bar{\alpha}\bar{\alpha}_i + (\bar{\alpha}\bar{\sigma} - (\bar{\alpha}\bar{\sigma} - 2\bar{\alpha}))\bar{\alpha}_i = 2\bar{\alpha}\bar{\alpha}_i + 2\bar{\alpha}\bar{\alpha}_i = 4\bar{\alpha}\bar{\alpha}_i .$$

Consequently

$$\bar{\alpha}\bar{\alpha}_i = 0\ \ (i = 1, 2, \ldots).$$

So, recalling (2), we find

$$\bar{R} = \mathcal{L}a\,(\bar{\alpha}\bar{\tau},\ \bar{\alpha}\bar{\sigma} - 2\bar{\alpha}).$$

Since $\bar{F}\bar{R} = 0$, it follows $\bar{R}$ is of dimension at most two. Therefore $\bar{F}^2 \cap \bar{R}$ is also finite dimensional (actually of dimension one) i.e. $H_2(L, \mathbf{Q})$ is finite dimensional as claimed.

5.2. We begin the proof that L is not finitely presented by proving that L is metabelian i.e. L^2 is abelian.

Lemma C2. *L is metabelian; specifically $\mathcal{L}a\,((\ldots\underbrace{(at)\ldots)}_{i}t\ (i = 0, 1, \ldots))$ is abelian.*

Proof. Put

$$a_i = (\ldots\underbrace{(at)\ldots)}_{i}t\ \ (i = 0, 1, \ldots).$$

Then it is not hard to see that

$$L^2 = \mathcal{L}a(a_0, a_1, \ldots).$$

In order to prove L^2 abelian observe that, by the very definition of L, $a_0 a_i = 0$ for every i. Hence

$$a_1 a_i = (a_0 t) a_i = (a_i t) a_0 + (a_0 a_i) t = a_{i+1} a_0 + 0 = 0$$

and hence inductively

$$a_i a_j = 0 \text{ for every } i, j.$$

Thus L^2 is abelian as claimed.

Next we explicitly observc (we have already made tacit use of this in Sections 3 and 4) that if L is finitely presented then a finite subset of any given set of defining relations in terms of a finite system of generators suffice to define L (cf. e.g. Cohn [1, p. 154]). So if L is finitely presented it can be presented in the form

$$L = \langle a, s, t; a(\underbrace{(\ldots(at)\ldots)}_{i}t) = 0 \ (i = 1, 2, \ldots, l),\ st = 0,\ as = 2a\rangle$$

where l is a positive integer. It is not difficult to deduce that if M is the Lie subalgebra of L generated by a and t then M can be presented in the form

(3) $$M = \langle a, t; a(\underbrace{(\ldots(at)\ldots)}_{i}t) = 0 \ \ (i = 1, 2, \ldots, l)\rangle,$$

i.e. M is also finitely presented. Now by Lemma C2

$$a(\underbrace{(\ldots(at)\ldots)}_{i}t) = 0 \ \ (i = 1, 2, \ldots).$$

Hence we can also present M in the form

(4) $$M = \langle a, t; a(\underbrace{(\ldots(at)\ldots)}_{i}t) = 0 \ \ (i = 1, 2, \ldots)\rangle.$$

The upshot of these remarks is that the Lie algebra M as presented by (4) is finitely presented. In Lemma C3 below we shall prove that this is false and thereby complete the proof that L itself is not finitely presented.

Lemma C3. *M is not finitely presented.*

Proof. Let $R = \mathbf{Z}[x]$ be the polynomial ring over $\mathbf{Z}$, the ring of integers, in a single

indeterminate x, and consider the ring Λ of all 3×3 matrices over R. Put

$$\tilde{a} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \tilde{t} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and let $\tilde{M}$ be the Lie subalgebra of $(\Lambda, \circ)$ generated by $\tilde{a}$ and $\tilde{t}$. Let $\tilde{N}$ be the ideal of $\tilde{M}$ generated by $\tilde{a}$. It is easy to check that $\tilde{N}^2$ is infinite dimensional and that

(5) $$\tilde{N}^2 \tilde{M} = 0.$$

Moreover the elements

$$\hat{a} = \tilde{a} + \tilde{N}^2, \ \ \hat{t} = \tilde{t} + \tilde{N}^2 \ \ (\text{in } \tilde{M}/\tilde{N}^2)$$

satisfy the relations (cf. (3))

$$\hat{a}(\underbrace{(\ldots(\hat{a}\hat{t})\ldots)}_{i}\hat{t}) = 0 \ \ (i = 1, 2, \ldots).$$

It follows, without difficulty, that

(6) $$M \cong \tilde{M}/\tilde{M}^2.$$

Finally let F be the free Lie algebra on σ and τ and let R be the kernel of the homomorphism of F onto M defined by

$$\sigma \to a, \ \ \tau \to t.$$

If M is finitely presented then R is the ideal generated by a finite set. Hence R/FR is finite dimensional.

On the other hand let θ be the homomorphism of F onto $\tilde{M}$ defined by

$$\sigma\theta = \tilde{a}, \ \ \tau\theta = \tilde{t}.$$

Let S be the kernel of θ. Then

$$F/S \cong \tilde{M}.$$

Now in view of the isomorphism (6) the inverse image of $\tilde{N}^2$ under θ is simply R. It follows from (5) then that $FR \leqslant S$ and hence R/S is a quotient of R/FR. This means

$$R/S \cong \tilde{N}^2$$

is finite dimensional in contradiction to the remark made immediately before (5). This completes the proof of Lemma C3.

References

[1] P.M. Cohn, Universal Algebra (Harper and Row, 1965).
[2] Nathan Jacobson, Lie Algebras (Interscience Publishers, 1962).
[3] J. Knopfmacher, Homology and presentations of algebras, Proc. American Math. Soc. 66 (1967) 1424–1428.